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# AUTOSYNARTETIC SOLUTIONS OF DIFFERENTIAL EQUATIONS.*

By D. C. Lewis.[1]

1. **Introduction.** We are concerned with the differential system

$$dx/dt = f(t, x), \tag{1.1}$$

where $x$ and $f$ are $n$-vectors and $t$ is the scalar independent variable. We assume that $f$ is defined and of class $C'$ in a suitably chosen region, which we shall not need to specify in detail.

A simple and familiar example of the sort of thing we wish to study occurs when $f$ is periodic in $t$ with period $T$. This situation may be described by saying that the transformation

$$s = t + T, \qquad y = x, \tag{1.2}$$

takes the equation (1.1) into the form $y' = f(s, y)$ where the accent denotes differentiation with respect to $s$. This is just the original equation in a different notation. If, now, it is known that a particular solution $x(t)$ of (1.1) has the property that $x(0) = x(T)$, it is immediately obvious that $x(t)$ must be periodic with period $T$; in other words, $x(t)$ must satisfy a functional equation,

$$x(t + T) = x(t), \tag{1.3}$$

at least, if $x(t)$ can be defined for all values of $t$.

Suppose now that (1.1) is carried into itself by a more complicated transformation than (1.2), say by the transformation,

$$s = P(t, x), \qquad y = h(t, x), \tag{1.4}$$

where $P$ is a scalar and $h$ a vector and both are of class $C'$ in $t$ and $x$. If, now, it is known that a certain solution $x(t)$ has the property that $h(0, x(0))$

* Received April 18, 1960.

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$=x(P(0,x(0)))$ it will turn out that $x(t)$ must satisfy the functional equation,

$$x(P(t,x(t)))=h(t,x(t)), \tag{1.5}$$

at least, if $x(t)$ can be defined for all values of $t$.

Evidently, if $P(t,x)=t+T$ and $h(t,x)=x$, (1.4) and (1.5) reduce respectively to (1.2) and (1.3); so that the situation in which $f(t,x)$ is periodic in $t$ is indeed a special case of what we wish to consider.

An important problem concerning the simpler situation is the perturbation of periodic solutions with respect to a parameter, a problem first considered by Poincaré. We shall show that the main features of this theory of Poincaré and his followers may be extended to cover the perturbation of a solution satisfying (1.5). Such solutions will be called autosynartetic; and we shall present theorems about the *degeneracy* of an autosynartetic solution, about the associated so-called *bifurcation equations*, about the influence of certains kinds of *first integrals* on the degeneracy and on the bifurcation equations, and about phenomena associated with the presence of certain kinds of *continuous groups* (*or semi groups*) *of transformations* which take (1.1) into itself.

For instance, when the system (1.1) is autonomous (i.e. when $f(t,x)=f(x)$ is independent of $t$) and when therefore we may imbed the transformation (1.2) in the continuous group of transformations,

$$s=t+T+\lambda, \qquad y=x, \tag{1.6}$$

(where $\lambda$ is the parameter of the group) in such a manner that (1.6) equally with (1.2) takes (1.1) into itself, the well known fact, that any non-constant periodic solution of $dx/dt=f(x)$ must be degenerate, is just a simple special case of one of our theorems on general autosynartetic solutions.

Our theories, in addition to applying to the entirely new subject of general autosynartetic solutions, may even yield a few new results about the simpler cases already studied. Thus, for instance, to the best of the author's knowledge, Theorems 8.4, 8.5, and 8.6, specialized to the periodic case, have not appeared in fully developed form in the literature, although partial results along these lines are given in [4] (cf. the References at the end of the paper). Again, in some special non-autonomous periodic case, it might be possible to imbed (1.2) in some 1-parameter transformation group,

$$s=P(t,x,\lambda), \qquad y=h(t,x,\lambda),$$

with $P(t,x,0)=t+T$ and $h(t,x,0)=x$. Our theory then says that any periodic solution of (1.1) must be degenerate, at least, if we impose one

further mild condition on the periodic solution in question, which takes the place of the condition mentioned above in the autonomous case that the solution should not be a constant.

We shall also discuss necessary and sufficient conditions that (1.4) should transform (1.1) into itself, using a definition which may strike the reader as a little peculiar, since it does not require (1.4) to possess an inverse. This definition was adopted, since in the major part of the paper we never need to assume the existence of an inverse to (1.4). We shall show that infinitely many such transformations always exist, in fact, even when $P(t,x)$, as well as $f(t,x)$, is prescribed. These transformations are, however, determined by the integration of a system more complicated than (1.1) and hence cannot be expected to be of any real help in any qualitative discussion of the solutions of (1.1). Such help is to be expected only when a particular transformation is more or less obvious from the structure of the equations. Thus, for example, the equations of celestial mechanics admit both rotational and translational symmetry. It often happens, under such circumstances, that the functional equation (1.5) merely expresses the fact that $x(t)$ is in some sense a periodic solution, when suitable coordinates are employed (cf. "les trois sortes de solutions périodiques" of Poincaré [5], vol. 1, pp. 95-97). At other times it gives us periodic solutions possessing certain special properties of symmetry. An indication of how this may occur is as follows:

Suppose that, in (1.4), $P(t,x)=t+T$, while $y=h(t,x)=h_1(x)$ is independent of $t$ and generates a finite cyclic group of transformations of order $k$. Then the functional equation (1.5) implies that

$$x(t+mT)=h_m(x(t)),$$

where $h_m$ denotes the $m$-th iterate of $h_1$. Since $h_k$ is the identity transformation, we see at once that, under present hypotheses, our functional equation expresses the fact that $x(t)$ is a certain special kind of periodic solution with period $kT$.

Another possibility may arise if $y=h_1(x)$ generates an infinite free group, such that, for any positive $\epsilon$, we shall have $|h_m(x)-x|<\epsilon$ for infinitely many values of $m$. We then get a recurrent solution, or, if $h_m(x)$ recurs to an approximation of the identity in a suitably uniform manner, we would get an almost periodic solution.

In spite of these trivialities in the presently contemplated applications, the transformations considered in this paper can be of a much more general type, well deserving serious study for their own sake.

## 2. Some elementary theorems on the transformations of differential equations.

DEFINITION 2.1. *To say that* (1.4) *transforms the system* (1.1) *into the system,*

$$dy/ds = g(s, y), \tag{2.1}$$

*means that to every solution* $x(t)$ *of the system* (1.1) *there corresponds a solution* $y(s)$ *of* (2.1) *such that*

$$y[P(t, x(t))] = h[t, x(t)]. \tag{2.2}$$

Notice that this definition does not require (1.4) to have an inverse, although certainly one would ordinarily expect an inverse to exist in cases of principal importance.

THEOREM 2.1. *If* (1.4) *transforms* (1.1) *into* (2.1), *the functions* $h(t, x)$ *and* $P(t, x)$ *must satisfy the vector partial differential equation,*

$$\begin{aligned} &h_t(t, x) + h_x(t, x)f(t, x) \\ &\qquad = g[P(t, x), h(t, x)][P_t(t, x) + P_x(t, x)f(t, x)], \end{aligned} \tag{2.3}$$

*at every point of the common domain of definition of* $f$, $P$, *and* $h$.

*Proof.* Differentiate the identity (2.2) with respect to $t$ and use the fact that $dy(s)/ds = g(s, y(s))$ with $s = P(t, x(t))$. We also use the facts that $dx(t)/dt = f(t, x(t))$ and consequently that

$$ds/dt = P_t(t, x(t)) + P_x(t, x(t))f(t, x(t)).$$

The result is

$$\begin{aligned} &g[P(t, x(t)), y[P(t, x(t))]] \\ &\qquad [P_t(t, x(t)) + P_x(t, x(t))f(t, x(t))] \equiv h_t(t, x(t)) + h_x(t, x(t))f(t, x(t)). \end{aligned}$$

Finally we notice that, by the existence theorem for the system (1.1), there is a solution through every point of the space where $f$ is defined. Hence the last identity in $t$, based on an arbitrary solution $x(t)$ of (1.1), becomes an identity in $t$ and $x$, if we eliminate $y$ with the help of (2.2) and then replace $x(t)$ simply by $x$.

THEOREM 2.2. *If* $P(t, x)$ *and* $f(t, x)$ *have the property that*

$$P_t(t, x) + P_x(t, x)f(t, x) \neq 0 \tag{2.4}$$

*at every point of their common domain of definition and if $h(t,x)$ is any vector function satisfying (2.3), then (1.4) transforms (1.1) into (2.1).*

*Proof.* Let $x(t)$ be any solution of (1.1). Then by (2.4) we have

$$(d/dt)[P(t,x(t))] = P_t(t,x(t)) + P_x(t,x(t))f(t,x(t)) \neq 0.$$

Hence by the implicit function theorem, the equation

$$s = P(t, x(t)) \tag{2.5}$$

may be solved for $t$ in terms of $s$. It is then easy to see from Definition 2.1 that it will be enough to prove that the vector function $y(s)$ defined as being the same as $h[t(s), x(t(s))]$ is a solution of (2.1). Remembering that $y(s(t)) = h(t, x(t))$, we see from (2.5) and (1.1) that $dy(s(t))/ds(t) = (dy/dt)/(ds/dt) = [h_t(t,x) + h_x(t,x)f(t,x)]\,[P_t(t,x) + P_x(t,x)f(t,x)]^{-1}$, where we have, of course, used $x$ as an abbreviation for $x(t)$. We thus obtain from (2.3) the result that

$$y'(s(t)) = g[P(t,x), h(t,x)] = g[s(t), y(s(t))].$$

Changing the independent variable from $t$ to $s$, we obtain the desired result.

Before passing to the next theorem, it is convenient to introduce the following further notation: Let

$$x = \xi(t, t_0, x_0) \tag{2.6}$$

be the solution of (1.1) such that

$$\xi(t_0, t_0, x_0) = x_0. \tag{2.7}$$

Let $x = \xi(t, t_0, x_0)$ together with $u = U(t, t_0, x_0, u_0)$ be the solution of the following enlarged system of order $2n$:

$$dx/dt = f(t,x), \tag{2.8a}$$

$$du/dt = G(t,x,u), \tag{2.8b}$$

where $G(t,x,h)$ is an abbreviation for $g[P(t,x),h][P_t(t,x) + P_x(t,x)f(t,x)]$, so that consequently the partial differential equation (2.3) appears in the abbreviated form

$$h_t(t,x) + h_x(t,x)f(t,x) = G[t,x,h(t,x)]. \tag{2.9}$$

The function $U(t, t_0, x_0, u_0)$ is to satisfy the condition

$$U(t, t_0, x_0, u_0) = u_0. \tag{2.10}$$

The fact, that such functions $\xi$ and $U$, affording a solution of (2.8) and satisfying the initial conditions indicated by (2.7) and (2.10), actually exist, is, of course, a consequence of the known theory of ordinary differential equations.

THEOREM 2.3. *The n-vector function*

$$h(t,x) = U[t, t_0, \xi(t_0, t, x), H(\xi(t_0, t, x))] \tag{2.11}$$

*satisfies the partial differential equation* (2.9) (*which is the same as* (2.3)), *together with the initial condition,*

$$h(t_0, x) = H(x). \tag{2.12}$$

Here it is understood that $H(x)$ is an arbitrary $n$-vector function of class $C'$.

The proof of this theorem may be left to the reader, since a theory for the equation (2.9) in which $h$ is a vector may be formulated exactly as if $h$ were a scalar. Such a theory in the scalar case is given by Kamke [3], vol. 2, pp. 40-42. It may also be remarked that the verification of (2.12) is immediate with the help of (2.7) and (2.10), and that the verification of (2.9), although somewhat more complicated, may be carried out in a straightforward manner by differentiation of (2.11) and by use of well known properties of the functions $\xi$ and $U$.

**3. The transformation of a system of differential equations into itself.** We continue the discussion initiated in the preceding section only for the special case in which

$$g(t,x) \equiv f(t,x). \tag{3.1}$$

*We also shall assume throughout the rest of the paper that the inequality* (2.4) *holds.* With this understanding the following theorem is an immediate corollary of Theorems 2.1 and 2.2).

THEOREM 3.1. *A necessary and sufficient condition that* (1.4) *transform* (1.1) *into itself is that*

$$h_t(t,x) + h_x(t,x)f(t,x) = f[P(t,x), h(t,x)][P_t(t,x) + P_x(t,x)f(t,x)] \tag{3.2}$$

Similarly specializing Theorem 2.3 by (3.1), and using Theorem 3.1, we may state the obvious

THEOREM 3.2. *When $P(t,x)$ and $f(t,x)$ are given, $h$ may be found in infinitely many ways in such a manner that* (1.4) *transforms* (1.1) *into itself. In fact we may assign $h(t_0,x)$ arbitrarily for any fixed $t_0$.*

DEFINITION 3.1. *If* (1.4) *transforms* (1.1) *into itself, and if* $x(t)$ *is any solution of* (1.1), *the solution* $y(t)$ *of* (1.1), (*possibly not distinct from* $x(t)$), *which is such that*

$$y[P(t,x(t))] \equiv h(t,x(t)), \tag{3.3}$$

*is said to be synartetic to* $x(t)$ *under the transformation* (1.4). *If* $y(t) \equiv x(t)$, *we shall term* $x(t)$ *an autosynartetic solution of* (1.1) *under* (1.4).

Notice that, under the assumptions of this definition, the existence of the $y(t)$ mentioned above is *guaranteed* by Definition 2.1.

THEOREM 3.3. *Assuming that* (1.4) *transforms* (1.1) *into itself, a necessary and sufficient condition that a solution* $y(t)$ *be synartetic to a solution* $x(t)$ *of* (1.1) *under* (1.4) *is that*

$$y[P(t_0,x(t_0))] = h(t_0,x(t_0)) \tag{3.4}$$

*for any fixed* $t_0$.

*Proof.* The necessity of (3.4) is obvious, since we may substitute $t_0$ for $t$ in (3.3).

The sufficiency follows by an easy argument based, firstly, on the existence of a solution synartetic to a given solution, $x(t)$, as guaranteed above, and, secondly, on the uniqueness theorem for equation (1.1).

By considering the special case $y = x$, we get the following obvious corollary to Theorem 3.3, to which we must frequently refer in the later sections of this paper.

THEOREM 3.4. *Assuming that* (1.4) *transforms* (1.1) *into itself, a necessary and sufficient condition that a solution* $x(t)$ *of* (1.1) *should be autosynartetic under* (1.4) *is that*

$$x[P(t_0,x(t_0))] = h(t_0,x(t_0)) \tag{3.5}$$

*for any fixed* $t_0$.

In the next section we tacitly assume the validity of

THEOREM 3.5. *Let* $x(t,\alpha)$ *be a family of solutions of* (1.1) *which is continuous and continuously differentiable with respect to the parameter* $\alpha$. *Let* $y(t,\alpha)$ *be the family of solutions of* (1.1) *synartetic under* (1.4) *to* $x(t,\alpha)$. *Then* $y(t,\alpha)$ *is also continuous and continuously differentiable.*

*Proof.* We may, because of (2.4), solve the equation, $s = P(t,x(t,\alpha))$

for $t=t(s,\alpha)$. Hence the solution $y(t,\alpha)$, synartetic to $x(t,\alpha)$, is given by the formula, $y(s,\alpha)=h[t(s,\alpha),x(t(s,\alpha),\alpha)]$. If $h$, $P$, and $x$ are of class $C'$, so is $t(s,\alpha)$, and hence eventually $y(s,\alpha)$.

**4. Properties of the variational system and of its adjoint.** Let us consider an arbitrary solution $x=\phi(t)$ of (1.1). The equations of variation based on this solution are

$$d\xi/dt=A(t)\xi, \tag{4.1}$$

where

$$A(t)=f_x(t,\phi(t)). \tag{4.2}$$

The equations of variation have the following familiar fundamental property: If $x=x(t,\alpha)$ is a family of solutions of (1.1) depending on a parameter $\alpha$ in any manner and such that $x(t,0)\equiv\phi(t)$, then $\xi(t)=\partial x(t,0)/\partial\alpha$ is a solution of (4.1).

Now let us suppose that (1.4) transforms (1.1) into itself. Let us denote the synartetic solution of $x(t,\alpha)$ by $y(t,\alpha)$. Then $\eta(t)=\partial y(t,0)/\partial\alpha$ will be a solution of the system $d\eta/dt=A^*(t)\eta$ where $A^*(t)=f_x(t,\phi^*(t))$, in which $\phi^*(t)=y(t,0)$ is of course synartetic to $x(t,0)=\phi(t)$. Consequently, if $\phi(t)$ is *auto*-synartetic, we shall have $\phi^*(t)=\phi(t)$ and $A^*(t)\equiv A(t)$; so that $\eta$ satisfies the same system (4.1) as $\xi$.

We now proceed to compute the relationship between $\xi$ and $\eta$. Since $y(t,\alpha)$ is synartetic to $x(t,\alpha)$, we know that $y(s,\alpha)$ is obtained from the equations,

$$y=h(t,x(t,\alpha)) \tag{4.3}$$

$$s=P(t,x(t,\alpha)) \tag{4.4}$$

by the elimination of $t$. Hence

$$\delta y/\delta\alpha=\partial y/\partial\alpha+(\partial y/\partial t)(\delta t/\delta\alpha) \tag{4.5}$$

where partial derivatives obtained under the assumption that $\alpha$ and $t$ are independent variables are denoted by $\partial$, while those obtained under the assumption that $\alpha$ and $s$ are independent variables are denoted by $\delta$. From (4.3) we obtain

$$\begin{aligned}&\partial y/\partial\alpha=h_x(t,x(t,\alpha))(\partial x/\partial\alpha)\quad\text{and}\\&\partial y/\partial t=h_t(t,x(t,\alpha))+h_x(t,x(t,\alpha))f(t,x(t,\alpha)),\end{aligned} \tag{4.6}$$

where we have simplified the last expression by using the fact that $x(t,\alpha)$

is a solution of (1.1). We further transform the last expression with the help of (3.2), thus obtaining

$$(4.7)\quad \partial y/\partial t = f[P(t, x(t, \alpha)), h(t, x(t, \alpha))]\,[P_t(t, x(t, \alpha)) + P_x(t, x(t, \alpha))f(t, x(t, \alpha))].$$

We next compute $\partial t/\partial\alpha$ from (4.4) as follows:

$$0 = \delta s/\delta\alpha = [P_t(t, x(t, \alpha)) + P_x(t, x(t, \alpha))f(t, x(t, \alpha))](\delta t/\delta\alpha) + P_x(t, x(t, \alpha))(\partial x/\partial\alpha).$$

Hence

$$\delta t/\delta\alpha = -(\partial x/\partial\alpha)P_x(t, x(t, \alpha))[P_t(t, x(t, \alpha)) + P_x(t, x(t, \alpha))f(t, x(t, \alpha))]^{-1}.$$

Using this last result together with (4.6) and (4.7) we see from (4.5) that

$$\delta y/\delta\alpha = [h_x(t, x(t, \alpha)) - f[P(t, x(t, \alpha)), h(t, x(t, \alpha))]^{\circ}P_x(t, x(t, \alpha))](\partial x/\partial\alpha).$$

Here, and throughout the remainder of the paper, $f^{\circ}P_x$ denotes the matrix of order $n$ the element in whose $i$-th row and $j$-th column is the product of the $i$-th component of $f$ by the $j$-th component of $P_x$. This is to be contrasted with the scalar product $P_xf$ used also very frequently in this paper.

Setting $\alpha = 0$, $x(t, 0) = \phi(t)$, $\delta y(s, 0)/\delta\alpha = \eta(s)$ and $\partial x(t, 0)/\partial\alpha = \xi(t)$, we obtain

$$(4.8)\qquad \eta(s) = B(t)\xi(t)$$

where the matrix $B(t)$ is defined by

$$(4.9)\qquad B(t) = h_x(t, \phi(t)) - f[P(t, \phi(t)), h(t, \phi(t))]^{\circ}P_x(t, \phi(t)),$$

or, since $\phi(t)$ is autosynartetic, by

$$(4.9\text{ alt.})\quad B(t) = h_x(t, \phi(t)) - f[P(t, \phi(t)), \phi(P(t, \phi(t)))]^{\circ}P_x(t, \phi(t)),$$

and where the $s$ of (4.8) is related to the $t$ by means of (4.4) with the $\alpha = 0$, in other words by

$$(4.10)\qquad s = P(t, \phi(t)) = p(t).$$

We interpret and summarize the main results of this discussion in the following:

THEOREM 4.1. *The variational system of* (1.1) *based on an autosynartetic solution* $\phi(t)$ *of* (1.1) *under* (1.4) *is transformed into itself by means of the transformation*

$$(4.11)\qquad s = p(t), \qquad \eta = B(t)\xi,$$

*where the scalar function* $p(t)$ *and the matrix function* $B(t)$ *are given respectively by* (4.10) *and by* (4.9).

*Proof.* According to Definition 2.1, we need only to show that to every *arbitrary* solution $\xi(t)$ of (4.1) there corresponds a solution $\eta(t)$ of (4.1) such that $\eta(p(t)) = B(t)\xi(t)$, or, in other words, such that (4.8) holds with $s$ given by (4.10). Our previous discussion therefore yields a complete proof, as soon as it is established that an *arbitrary* solution $\xi(t)$ can be represented in the form $\partial x(t,0)/\partial\alpha$ used above. This is an elementary detail which we leave to the reader.

An almost immediate corollary of Theorem 4.1 is

THEOREM 4.2. *Between the matrices* $A(t)$, $B(t)$, *and the scalar* $p(t)$ *there is the following relationship:*

$$dB(t)/dt + B(t)A(t) - A(p(t))B(t)(dp(t)/dt) \equiv 0. \tag{4.12}$$

*Proof.* Since (4.11) transforms (4.1) into itself we may apply the partial differential equation (3.2), with appropriate changes of notation, to the present linear situation. In this way we obtain the result that $(dB(t)/dt)\xi + B(t)A(t)\xi = A(p(t))B(t)\xi(dp(t)/dt)$. But, since this is an identity in the vector $\xi$, we obtain (4.12) at once.

It is also possible to verify (4.12) directly from the definitions of $A(t)$, $B(t)$, and $p(t)$ and from the fact that $h$ satisfies (3.2). But the calculation is rather tedious. From such an alternative proof of Theorem 4.2, we also have an alternative proof for Theorem 4.1, which follows from Theorem 4.2 with the help of Theorem 3.1.

At this stage, it is convenient to introduce the function $q(t)$ which is the inverse of $p(t)$, so that

$$p(q(t)) = q(p(t)) = t. \tag{4.13}$$

This is legitimate, since by (4.10) and (2.4),

$$\dot{p}(t) = P_t(t,\phi(t)) + P_x(t,\phi(t))f(t,\phi(t))$$

is never zero. We also introduce the matrix $C(t)$ defined by

$$C(t) = B(q(t))'. \tag{4.14}$$

The accent is used here and in the remainder of the paper to denote the transpose of a matrix.

THEOREM 4.3. *The linear system*

$$d\xi/dt = -A(t)'\xi \quad or \quad d\xi/dt = -\xi A(t), \tag{4.15}$$

*which is the so-called adjoint to the system* (4.1), *is transformed into itself by means of the transformation,*

$$s = q(t), \qquad \eta = C(t)\xi. \tag{4.16}$$

*Proof.* From Theorem 3.1, as in the proof of the previous theorem, we find that a necessary and sufficient condition that (4.5) be transformed into itself by (4.16) is the validity of the identity

$$dC(t)/dt - C(t)A(t)' + A(q(t))'C(t)(dq(t)/dt) \equiv 0. \tag{4.17}$$

If we replace $t$ by $p(t)$ and then multiply by $dp/dt$ we get the equivalent identity,

$$[dC(p(t))/dp](dp(t)/dt) - C(p(t))A(p(t))'(dp(t)/dt)$$
$$+ A[q(p(t))]'C(p(t))(dq(p(t))/dp)(dp(t)/dt) \equiv 0.$$

This, with the help of (4.13) and (4.14), is seen to be equivalent to

$$dB(t)'/dt - B(t)'A(p(t))'(dp(t)/dt) + A(t)'B(t)' \equiv 0.$$

But this last identity is merely the transpose of (4.12). Thus we have shown that (4.17) is equivalent to the already established identity (4.12).

THEOREM 4.4. *In order that a solution* $\xi(t)$ *of the variational equations* (4.1) *be autosynartetic under the transformation* (4.11) *it is necessary and sufficient that it satisfy the "boundary conditions,"*

$$\xi(T) = B\xi(0), \tag{4.18}$$

*where*

$$T = p(0) \quad and \quad B = B(0). \tag{4.19}$$

*Proof.* This is a special case of Theorem 3.4 applied to the linear system under (4.11) instead of to the general system (1.1) under (1.4). And, in making this application, we take $t_0 = 0$.

THEOREM 4.5. *In order that a solution* $\xi(t)$ *of the adjoint system* (4.15) *be autosynartetic under the transformation* (4.16) *it is necessary and sufficient that it satisfy the boundary conditions*

$$\xi(T)B = \xi(0) \tag{4.20}$$

*where* $T$ *and* $B$ *are given by* (4.19).

*Proof.* Again appealing to Theorem 3.4, we find that a necessary and sufficient condition that $\xi(t)$ be autosynartetic under (4.16) is that $\xi[q(T)] = C(T)\xi(T)$. In obtaining this result we take the $t_0$ of Theorem 3.4 to be $T$. From (4.13) and (4.14), we find that $q(T) = 0$ and that $C(T) = B'$, whence it appears that (4.20) is an equivalent condition.

Whenever we refer to autosynartetic solutions of the variational system or its adjoint, we always mean autosynarteticity under (4.11), in the case of the variational system, and under (4.16) in the case of the adjoint system. Thus, according to Theorems 4.4 and 4.5, autosynartetic solutions of the variational equations are those which satisfy the boundary conditions (4.18) and autosynartetic solutions of the adjoint equations are those which satisfy the boundary conditions (4.20). With this understanding we have

THEOREM 4.6. *The maximum number of solutions in any set of linearly independent autosynartetic solutions of the variational equations* (4.1) *is equal to the maximum number of solutions in any set of linearly independent autosynartetic solutions of the adjoint system* (4.15).

*Proof.* Let the $n \times n$ matrix $X(t)$ satisfy $dX/dt = A(t)X$ and $\det X(0) \neq 0$. Hence $\det X(t) \neq 0$. Corresponding to any arbitrary solution $\xi(t)$ of the variational equations, there exists an $n$-vector $c$ such that $\xi(t) = X(t)c$, and, by Theorem 4.4, this is autosynartetic under (4.11) if and only if $X(T)c = BX(0)c$, which is equivalent to $X(T)^{-1}[X(T) - BX(0)]c = 0$. *Hence the number of linearly independent autosynartetic solutions of the variational equations is equal to the number of linearly independent relationships between the columns of the matrix* $X(T)^{-1}[x(T) - BX(0)]$.

Now, if we set $Y(t)' = X(t)^{-1}$, it is well known (and indeed easy to see by differentiation of $Y(t)'X(t) = I$) that $dY/dt = -A(t)'Y$ and $\det Y(t) \neq 0$. Hence, corresponding to any arbitrary solution $\xi_1(t)$ of the adjoint system, there exists an $n$-vector $c_1$ such that $\xi_1(t) = Y(t)c_1$, and, by Theorem 4.5, this is autosynartetic under (4.16) if and only if $[Y(T)c_1]B = Y(0)c_1$ or $c_1Y(T)'B = c_1Y(0)'$. In terms of $X(t)$ this condition appears (after some simple manipulations) in the equivalent form $c_1X(T)^{-1}[X(T) - BX(0)] = 0$. *Hence the number of linearly independent autosyartetic solutions of the adjoint system is equal to the number of linearly independent relationships between the rows of the matrix* $X(T)^{-1}[X(T) - BX(0)]$.

The theorem follows at once from the above two italicized sentences.

**5. Synartetic first integrals and so-called degeneracy.** First integrals are defined in the usual way; namely a scalar function $J(t, x)$ is called a first

integral of (1.1), if its value is independent of $t$ whenever $x$ is replaced by any solution $x(t)$ of (1.1). It is both obvious and well known that a necessary and sufficient condition that $J$ be a first integral is that

$$(5.1) \qquad J_t(t,x) + J_x(t,x)f(t,x) \equiv 0,$$

at least, if $J$ is of class $C'$.

If (1.1) is transformed into itsef by (1.4), then with every solution $x(t)$ of (1.1) there is associated a synartetic solution $y(t)$. Hence, if $J(t,x)$ is a first integral of (1.1), we see that $J(s,y(s))$ must be independent of $s$. Hence, setting $s = P(t,x(t))$ and remembering that then $y(s) = h(t,x(t))$ by definition of a synartetic solution, we see that $J[P(t,x(t)),h(t,x(t))]$ must be independent of $t$ no matter with what solution $x(t)$ we may be dealing. In other words, we have established

THEOREM 5.1. *If $J(t,x)$ is a first integral of* (1.1) *and if* (1.1) *is transformed into itself by* (1.4), *then $J[P(t,x),h(t,x)]$ is also a first integral of* (1.1).

DEFINITION 5.1. *Under the assumptions of Theorem* 5.1, *the first integral $J[P(t,x),h(t,x)]$ is said to be synartetic to the first integral $J(t,x)$; and, if it happens that*

$$(5.2) \qquad J[P(t,x),h(t,x)] \equiv J(t,x),$$

*we shall say that the first integral $J(t,x)$ is autosynartetic under* (1.4).

THEOREM 5.2. *Let $\phi(t)$ be an autosynartetic solution of* (1.1) *under* (1.4). *Let $J(t,x)$ be an autosynartetic first integral of* (1.1) *also under* (1.4). *Then $\xi(t) = J_x(t,\phi(t))$ is an autosynartetic solution of the adjoint system* (4.15) *of the equations of variation* (4.1).

*Proof.* Differentiating (5.2) with respect to $x$ and then setting $x = \phi(t)$ and $p(t) = P(t,\phi(t))$, we find that

$$J_t[p(t),h(t,\phi(t))]P_x(t,\phi(t)) + J_x[p(t),h(t,\phi(t))]h_x(t,\phi(t)) \equiv J_x(t,\phi(t)).$$

But according to (5.1) we know that

$$J_t[p(t),h(t,\phi(t))] = -J_x[p(t),h(t,\phi(t))]f[p(t),h(t,\phi(t))].$$

Hence, eliminating $J_t[p(t),h(t,\phi(t))]$ and remembering that $h(t,\phi(t)) = \phi(p(t))$, since $\phi$ is autosynartetic, we find that

$$J_x[p(t),\phi(p(t))]\,[h_x(t,\phi(t)) - f[p(t),h(t,\phi(t))]^0 P_x(t,\phi(t))] = J_x(t,\phi(t)).$$

Thus, remembering that $p(t) = P(t, \phi(t))$ and $\xi(t) = J_x(t, \phi(t))$, we find from (4.9) that $\xi(p(t))B(t) = \xi(t)$. Setting $t = 0$, we find from (4.19) that $\xi(t)$ satisfies the boundary condition (4.20). It remains to show that $\xi(t)$ satisfies the adjoint system (4.15). Since this may be done as in the periodic special case, we leave this part of the proof to the reader. See Lewis [4], p. 542, lines 23-37, where, however, the notation as well as the context must be modified to fit present circumstances.

DEFINITION 5.2. *The degeneracy of an autosynartetic solution $\phi(t)$ of the system* (1.1) *is equal to the maximum number of solutions of the variational equations* (4.1) *in any set of linearly independent autosynartetic solutions of these equations or, what by Theorem* 4.6 *is the same thing, the maximum number of solutions of the adjoint equations* (4.15) *in any set of linearly independent autosynartetic solutions of the adjoint equations.*

If there are no autosynartetic solutions of (4.1) other than the trivial solution $\xi \equiv 0$, the degeneracy is of course 0 by the definition, and we then sometimes say that $\phi(t)$ is *non-degenerate.*

THEOREM 5.3. *If the system* (1.1) *admits $k$ "independent" autosynartetic first integrals, then the degeneracy of any autosynartetic solution $\phi(t)$ of* (1.1) *is at least $k$.*

Here the hypothesis that the $k$ first integrals should be independent is interpreted as meaning that the rank of the jacobian matrix of the $k$ integrals with respect to the components of $x$ should be $k$ when $x = \phi(t)$. This insures the existence of $k$ linearly independent autosynartetic solutions of the adjoint system as indicated in Theorem 5.2, so that no further proof is needed for Theorem 5.3.

THEOREM 5.4. *If* (1.1) *admits a $k$-parameter family of autosynartetic solutions, the degeneracy of any one of the solutions imbedded in the family is at least $k$.*

*Proof.* Suppose the $k$-parameter family of autosynartetic solutions of (1.1) is represented by $x = x(t, c)$, where $c$ is a $k$-vector and $x(t, 0) = \phi(t)$. Then the $k$ columns of the $nxk$ matrix $x_c(t, 0)$ are obviously solutions of the variational equations (4.1) based on $\phi(t)$. Moreover, since $x(t, c)$ for each fixed $c$ is given as autosynartetic under (1.4), we have from Theorem 3.4

$$x[P(0, x(0, c)), c] = h(0, x(0, c)).$$

Differentiating with respect to $c$ and then setting $c=0$ and also using the fact that $x(t,c)$ is a solution of (1.1) we obtain

$$f[P(0,\phi(0)),\phi[P(0,\phi(0))]]^{0}P_x(0,\phi(0))x_c(0,0)+x_c[P(0,\phi(0)),0]$$
$$=h_x(0,\phi(0))x_c(0,0),$$

which, because of (4.9 alt.), (4.10), and (4.19), can be written

$$x_c(T,0)=Bx_c(0,0).$$

Hence $x_c(t,0)$ satisfies the boundary conditions specified in Theorem 4.4. The fact that $x(t,c)$ is actually a $k$-parameter family means also that the rank of the matrix $x_c(t,c)$ is $k$. Thus, the $k$ columns of $x_c(t,0)$ yield $k$ linearly independent autosynartetic solutions of the variational equations, so that, by Definition 5.2, the degeneracy of $\phi(t)$ must be at least $k$.

THEOREM 5.5. *Assume that the set of all transformations which transform* (1.1) *into itself contains a continuous family $F$ of transformations of class $C'$ satisfying the following three conditions:*

P1. *Every transformation of $F$ commutes with* (1.4).

P2. *$F$ contains the identity transformation.*

P3. *There is a solution $\phi(t)$ which is autosynartetic under* (1.4) *but which is not autosynartetic under any transformation of $F$ close to the identity* (*except under the identity itself*).

*Then $\phi(t)$ has degeneracy* $\geqq 1$.

*Proof.* A transformation $S$ which transforms (1.1) into itself according to Definition 2.1 provides a mapping of the set of all solutions of (1.1) into itself. Namely $S$ maps an arbitrary solution $x$ of (1.1) onto the solution $y$ which is synartetic to $x$ under $S$. We express this mapping by writing $Sx=y$. Let $S^*$ denote the transformation (1.4) and let $S_\lambda$ denote a transformation of $F$, where $\lambda$ represents a set of values for the parameters of $F$. We may suppose by P2 that $S_0$ is the identity transformation.

By P3, we have $S^*\phi=\phi$. Hence $S_\lambda S^*\phi=S_\lambda\phi$. Whence, from P1, we obtain $S^*S_\lambda\phi=S_\lambda\phi$. Hence $S_\lambda\phi$ is autosynartetic under $S^*$. Hence by P2, $\phi=S_0\phi$ is imbedded in a continuous family of solutions $\{S_\lambda\phi\}$ each of which is autosynartetic under $S^*$, i.e. under (1.4). Because of P3, $S_\lambda\phi\neq\phi$ for $\lambda$ close to 0, except for $\lambda=0$. Hence the number of essential parameters in our family of autosynartetic solutions is surely greater than zero. Hence by Theorem 5.4 the degeneracy of $\phi$ is greater than zero, as we wished to prove.

In the above theorem the family $F$ does not necessarily form a group. In fact, our Definition 2.1 does not require the transformations which "transform (1.1) into itself" to have inverses: In fact, if $\{x\}$ is the set of all solutions $x$ of (1.1) and if $S$ is a transformation transforming (1.1) into itself, the set $\{Sx\}$ of all synartetic solutions under $S$ can be a proper subset of $\{x\}$.

If, however, the transformation $S^*$ (i.e. (1.4)) is an element of a full differentiable continuous group $G$ of transformations taking (1.1) into itself, we know from the theory of Lie that $G$ possesses a one-parameter continuous abelian subgroup $F$ also containing $S^*$. Hence the conditions P1 and P2 are automatically satisfied in this case. This is the situation, mentioned in Section 1, which occurs in the classical theory of periodic solutions of autonomous systems.

**6. Perturbation of autosynartetic solutions.** We now suppose that the system (1.1) contains a parameter $\mu$ and that the same may also be true of the transformation (1.4). We therefore write

$$dx/dt = f(t, x, \mu) \tag{6.1}$$

instead of (1.1), and

$$s = P(t, x, \mu), \qquad y = h(t, x, \mu) \tag{6.2}$$

instead of (1.4). We suppose that the dependence of $f$, $P$, and $h$ on $\mu$ is of class $C'$ and that (6.1) is transformed into itself by the transformation (6.2) for each value of $\mu$, such that $|\mu| < \beta$.

THEOREM 6.1. *Let $\phi(t)$ be a nondegenerate autosynartetic solution of (6.1) under the transformation (6.2) when $\mu = 0$. Then there exists a positive number $\beta^* \leqq \beta$, such that (6.1), for $|\mu| < \beta^*$, admits an autosynartetic solution $x(t, \mu)$ under the transformation (6.2). Moreover the dependence of $x(t, \mu)$ on $\mu$ is of class $C'$ and $x(t, \mu) \to \phi(t)$ as $\mu \to 0$, and $x(t, \mu)$ is the only autosynartetic solution of (6.1) in a suitably chosen neighborhood of $\phi(t)$.*

*Proof.* According to Theorem 3.4 (with $t_0 = 0$), a necessary and sufficient condition that a solution $x(t, \mu)$ be autosynartetic under (6.2) is that

$$x[P(0, x_0, \mu), \mu] - h(0, x_0, \mu) = 0, \tag{6.3}$$

where $x_0 = x_0(\mu) = x(0,\mu)$. Suppose $\psi(t, x_0, \mu)$ is the solution of (6.1) such that $\psi(0, x_0, \mu_0) = x_0$. Then (6.3) may be rewritten

$$(6.4) \qquad \psi[P(0,x_0,\mu), x_0, \mu] - h[0, x_0, \mu] = 0,$$

which we wish to solve for $x_0$ as a function of $\mu$ for $|\mu|$ sufficiently small, already knowing, of course, that (6.4) is satisfied by $x_0 = \phi(0)$ when $\mu = 0$. A straightforward calculation of the Jacobian $J$ of the left member of (6.4) with respect to $x_0$ at $\mu = 0$, $x_0 = \phi(0)$, leads, with the help of (4.19), (4.10) and (4.9 alt.) to the result, $J = X(T) - BX(0)$, where $T = P(0, \phi(0), 0) = p(0)$ and where $X(t) = \psi_{x_0}(t, \phi(0), 0)$ is a matrix solution of the variational equations (4.1), set up using $f(t,x,0)$ of (6.1) in place of the $f(t,x)$ of (1.1). If $\det J$ were zero, there would exist a vector $c \neq 0$ such that $[X(T) - BX(0)]c = 0$, and then according to Theorem 4.4, $\xi(t) = X(t)c$ would be a non-trivial autosynartetic solution of the variational equations, contrary to the hypothesis that $\phi(t)$ is non-degenrate. Thus $\det J \neq 0$, and applying the implicit function theorem, we write $x(t,\mu) = \psi(t, x_0(\mu), \mu)$. This $x(t,\mu)$ is easily seen to have the properties stated in the theorem.

Most of the details in this proof are left to the reader because of the similarity to the well known periodic special case.

**7. Lemmas on non-homogeneous linear systems.** In order to give a satisfactory analysis of the perturbation of autosynartetic solutions which are not non-degenerate, we cite the following lemma which is a slight variation of Lemma 1 in [4].

LEMMA 7.1. *Consider the linear differential system*

$$(7.1) \qquad d\xi/dt = A(t)\xi + f(t),$$

*where $\xi$ and $f$ are n-vectors and $A$ is an $n \times n$ matrix. $A$ and $f$ are known continuous functions of $t$, defined on the closed interval $\langle 0, T\rangle$ between 0 and $T$ ($T$ may be either positive or negative, but not zero). Let $X(t)$ be any $n \times n$ matrix, such that $dX/dt = A(t)X$ and $\det X(0) \neq 0$ (and hence also $\det X(t) \neq 0$ for any $t$ on $\langle 0, T\rangle$). Let $n - k$ denote the rank of the $n \times n$ matrix $BX(0) - X(T)$, where $B$ is a constant $n \times n$ matrix. Thus $k$ is the number of linearly independent solutions of the homogeneous system corresponding to (7.1) satisfying the "autosynartetic" boundary condition, $B\xi(0) = \xi(T)$.*

*Then there exist (independently of $f$) a $k \times n$ matrix function $\Xi(s)$ and an $n \times n$ matrix $G(t,s)$, both continuous, except that $G(t,s)$ possesses a finite jump at $t = s$, having the following properties:*

(I) *The system* (7.1) *possesses a solution satisfying the boundary condition* $B\xi(0)=\xi(T)$, *if and only if*

$$\int_0^T \Xi(s)f(s)\,ds=0. \tag{7.2}$$

(II) *If* (7.2) *is satisfied, the vector function,*

$$\xi(t)=\int_0^T G(t,s)f(s)\,ds$$

*is a solution of* (7.1) *and is, moreover, the only solution satisfying the boundary condition* $B\xi(0)=\xi(T)$ *which is orthogonal to every solution of the corresponding homogeneous system taken with the same boundary condition.*

(III) *Whether* (7.2) *is satisfied or not,* $\xi(t)$, *defined by* (7.3), *satisfies the boundary condition* $B\xi(0)=\xi(T)$.

(IV) *The rows of* $\Xi(s)$ *are orthogonal to the rows of* $G(t,s)$. *That is,*

$$\int_0^T G(t,s)\Xi(s)'\,ds=0. \tag{7.4}$$

(V) *The* $k\times k$ *matrix,*

$$C=\int_0^T \Xi(s)\Xi(s)'\,ds, \tag{7.5}$$

*is nonsingular.*

This Lemma may be regarded as well known. See for example [6], where however, the Lemma does not appear in exactly the desired form. In Lewis [4], pp. 537-540, will be found a complete proof [2] in the special case $B=I$. Only trivial modifications are needed in this proof to establish the lemma in its present form. Although we shall thus omit the proof of Lemma 7.1, we must record for future use certain supplementary facts.

Since $(n-k)$ is the rank of $BX(0)-X(T)$, there is a $k\times n$ matrix $\mathfrak{A}$ and an $n\times k$ matrix $\mathfrak{B}$, both of rank $k$, such that

$$\mathfrak{A}[BX(0)-X(T)]=0, \qquad [BX(0)-X(T)]\mathfrak{B}=0. \tag{7.6}$$

In terms of $\mathfrak{A}$, the $k\times n$ matrix function $\Xi(s)$ of the Lemma may be taken as

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[2] There is one formula in this proof which is in error. Namely the last formula on p. 539 should be $R(t)=\int_0^T G(t,s)\Xi(s)'C^{-1}ds$. This error fortunately does not effect the validity of the rest of the proof.

$$\Xi(s) = \mathfrak{A} X(T) X(s)^{-1}. \tag{7.7}$$

Cf. Lewis [4], p. 537, formula (2.9). The $k$ rows of $\Xi(t)$ are also seen to afford a full set of linearly independent autosynartetic solutions of the adjoint system (4.15). This is established with the help of (7.6), Theorems 4.5, 4.6, and the well known fact that the rows of $X(t)^{-1}$ satisfy the adjoint system.

Finally

$$\det\left[\int_0^T \mathfrak{B}' X(t)' X(t) \mathfrak{B}\, dt\right] \neq 0. \tag{7.8}$$

Cf. Lewis [4], p. 538, formula (2.15) and the accompanying discussion, in which it is shown that the matrix in question is positive definite.

LEMMA 7.2. *Under the same hypotheses as in the previous lemma and using the same notation, the following* $(n+k)\times(n+k)$ *matrix has a non-zero determinant:*

$$\mathfrak{M} = \left\| \begin{matrix} [BX(0) - X(T)] & \int_0^T X(T) X(s)^{-1} \Xi(s)' ds \\ \int_0^T \mathfrak{B}' X(t)' X(t)\, dt & \mathfrak{K} \end{matrix} \right\|$$

*where the block in the upper left hand corner (viz.* $BX(0) - X(T)$*) is an* $n \times n$ *matrix, while the block in the lower right hand corner (viz.* $\mathfrak{K}$*) is an arbitrary* $k \times k$ *matrix.*

*Proof.* Suppose $\det \mathfrak{M} = 0$. Then there would exist a linear relationship between the columns of $\mathfrak{M}$ with coefficients which would not all be zero. In other words, *there would exist an n-vector* $\beta$ *and a k-vector* $\alpha$*, whose components are not all zero such that*

$$[BX(0) - X(T)]\beta + \left[\int_0^T X(T) X(s)^{-1} \Xi(s)' ds\right]\alpha = 0, \tag{7.9}$$

$$\left[\int_0^T \mathfrak{B}' X(t)' X(t)\, dt\right]\beta - \mathfrak{K}\alpha = 0. \tag{7.10}$$

Consider now the $n$-vector function

$$\xi(t) = X(t)\beta - \int_0^t X(t) X(s)^{-1} \Xi(s)' \alpha\, ds. \tag{7.11}$$

This $\xi(t)$ clearly satisfies the equation (7.1) in the special case that $f(t) = -\Xi(t)'\alpha$. Moreover (7.9) expresses the fact that $\xi(t)$ satisfies the

boundary condition, $B\xi(0) = \xi(T)$. Hence Lemma 7.1 assures us, by means of (7.2), that

$$\int_0^T \Xi(s)\Xi(s)'\alpha\, ds = C\alpha = 0.$$

Referring now to (V) of Lemma 7.1, we conclude that $\alpha = 0$. Thus $\xi(t) = X(t)\beta$, whereas (7.10) expresses the fact that $\xi(t)$, which we already know satisfies the boundary condition, is also orthogonal to every solution, satisfying the same boundary condition, of the homogeneous equation $d\xi(t)/dt = A(t)\xi(t)$. In particular $\xi(t)$ must be orthogonal to itself. Hence it must vanish identically, i.e. $X(t)\beta = 0$, and since $X(t)$ is non-singular, we conclude that $\beta = 0$. With both $\alpha$ and $\beta$ necessarily reducing to zero, we have reached a contradiction of the above italicized statement resulting from the assumption that $\det \mathfrak{M} = 0$. This is all that is needed to establish the lemma.

**8. The bifurcation equations in the degenerate cases.** Suppose that $\phi(t)$ is an autosynartetic solution of (6.1) under the transformation (6.2) when $\mu = 0$. We consider the variational equation (4.1) with $A(t) = f_x(t, \phi(t), 0)$. We suppose that $\phi(t)$ has degeneracy $k$, which means that (4.1) has just $k$ linearly independent solutions satisfying the boundary conditions (4.18). In referring in this way to the material of Section 4, we mean that the $f(t,x)$, $h(t,x)$ and $P(t,x)$ of Section 4 are to be identified with the $f(t,x,0)$, $h(t,x,0)$, and $P(t,x,0)$ respectively of this Section or Section 6.

We define the $n \times n$ matrix $X(t)$ by the two conditions

$$dX(t)/dt = A(t)X(t), \qquad X(0) = I, \tag{8.1}$$

and then introduce the matrices $\mathfrak{A}$, $\mathfrak{B}$, $\Xi(t)$ as in Section 7.

THEOREM 8.1. *It is possible to define a unique n-vector function $x(t,c,\mu)$ and a k-vector function $\alpha(c,\mu)$ of class $C'$ for sufficiently small c and $|\mu|$, where c is a k-vector, in such a manner that the following four conditions are fulfilled:*

$$x_t(t,c,\mu) = f[t, x(t,c,\mu), \mu] - \Xi(t)'\alpha(c,\mu) \tag{8.2}$$

$$x[P(0, x(0,c,\mu), \mu), c, \mu] = h[0, x(0,c,\mu), \mu] \tag{8.3}$$

$$\int_0^T \mathfrak{B}'X(t)'[x(t,c,\mu) - \phi(t) - X(t)\mathfrak{B}c]\,dt = 0 \tag{8.4}$$

$$x(t,0,0)=\phi(t). \tag{8.5}$$

*Proof.* We introduce the $n$-vector function $\psi(t,x_0,\alpha,\mu)$ in such a manner that

$$x=\psi(t,x_0,\alpha,\mu) \tag{8.6}$$

is the solution of the differential system,

$$\dot{x}=f(t,x,\mu)-\Xi(t)'\alpha \tag{8.7}$$

which reduces to $x_0$, when $t=0$. That is,

$$\psi(0,x_0,\alpha,\mu)=x_0. \tag{8.8}$$

Other more or less obvious properties of $\psi$ are the following:

$$\psi(t,\phi(0),0,0)=\phi(t). \tag{8.9}$$

$$\psi_{x_0}(t,\phi(0),0,0)=X(t). \tag{8.10}$$

$$\psi_\alpha(t,\phi(0),0,0)=\sigma(t)=-\int_0^t X(t)X(s)^{-1}\Xi(s)'ds, \tag{8.11}$$

where $\sigma$ is an $n\times k$ matrix which vanishes when $t=0$ and satisfies the system $d\sigma/dt=A(t)\sigma-\Xi(t)'$. To give some indication of how these properties are derived, we remark that (8.9) follows from the uniqueness theorem for (8.7) and the definition of $\phi(t)$. We derive (8.10) when we substitute $\psi(t,x_0,\alpha,\mu)$ for $x$ in (8.7), differentiate with respect to $x_0$, set $x_0=\phi(0)$, $\mu=0$, $\alpha=0$, and use the definition of $A(t)=f_x(t,\phi(t),0)$ and (8.1). We get (8.11) in a similar manner by differentiting with respect to $\alpha$. The expression for $\sigma(t)$, of course, comes from the Lagrange result for "variation of parameters."

We next try to find $x_0$ and $\alpha$ as functions of $c$ and $\mu$ in such a manner that, when the functions are inserted in (8.6), $\psi(t,x_0(c,\mu),\alpha(c,\mu),\mu)$ reduces to the required $x(t,c,\mu)$ satisfying conditions (8.2), (8.3), (8.4), and (8.5). No matter how $x_0(c,\mu)$ and $\alpha(c,\mu)$ are chosen (8.2) is automatically satisfied because $\psi$ was defined to be a solution of (8.7); and, if furthermore we require that

$$x_0(0,0)=\phi(0), \tag{8.12}$$

$$\alpha(0,0)=0, \tag{8.13}$$

we see that (8.5) is also satisfied because of (8.9). The other two con-

ditions (8.3) and (8.4) lead to the following vector equations for the determination of $x_0$ and $\alpha$ as functions of $c$ and $\mu$.

$$(8.14) \qquad -\psi[P(0,\psi(0,x_0,\alpha,\mu),\mu),x_0,\alpha,\mu]+h[0,\psi(0,x_0,\alpha,\mu),\mu]=0.$$

$$(8.15) \qquad \int_0^T \mathfrak{B}'X(t)'[\psi(t,x_0,\alpha,\mu)-\phi(t)-X(t)\mathfrak{B}c]dt=0.$$

We note first of all that, consistently with (8.12) and (8.13), the last two equations (viz. (8.14) and (8.15)) are satisfied when $x_0=\phi(0)$, $\alpha=0$, $c=0$, and $\mu=0$. This is because (8.9) and the fact that $\phi[P(0,\phi(0),0)]=h[0,\phi(0),0]$, since $\phi$ is given as an autosynartetic solution of $\dot{x}=f(t,x,0)$ under the transformation $s=P(t,x,0)$, $y=h(t,x,0)$.

Hence, if we can show that the jacobian of the system furnished by (8.14) and (8.15) with respect to the components of $x_0$ and $\alpha$ does not vanish when $x_0=\phi(0)$, $\alpha=0$, $c=0$, and $\mu=0$, these equations, (8.14) and (8.15), by the implicit function theorem, effectively give the required functions $x_0(c,\mu)$ and $\alpha(c,\mu)$ for sufficiently small values of $c$ and $\mu$. Hence the proof of Theorem 8.1 will be complete as soon as we show that this jacobian is not 0.

Differentiating the left member of (8.14) with respect to $x_0$ and then setting $x_0=\phi(0)$, $\alpha=0$, $\mu=0$, gives with the help of (8.9) and (8.10) the following:

$$h_x[0,\phi(0),0]X(0)-\psi_t[P(0,\phi(0),0),\phi(0),0,0]^0P_x(0,\phi(0),0)-X(P(0,\phi(0),0)),$$

which, from the facts that $X(0)=I$, that $P(0,\phi(0),0)=T$ and that $\psi$ satisfies (8.7) and (8.9), can be written in the form,

$$[h_x[0,\phi(0),0]$$
$$-f[P(0,\phi(0),0),\phi(P(0,\phi(0),0)),0]^0P_x(0,\phi(0),0)]X(0)-X(T).$$

Hence, from (4.9 alt.) and (4.19), we find that the jacobian matrix of (8.14) with respect to the components of $x_0$ at the point in question is the $n\times n$ matrix $BX(0)-X(T)$, which is just the block of elements in the upper left hand corner of the matrix $\mathfrak{M}$ of Lemma 7.2.

Differentiating the left member of (8.14) with respect to $\alpha$, using (8.11) and evaluating at $x_0=\phi(0)$, $\alpha=0$, $\mu=0$, yields

$$\int_0^T X(T)X(s)^{-1}\Xi(s)'ds,$$

which is the block of elements in the upper right hand corner of $\mathfrak{M}$.

Differentiating the left member of (8.15) with respect to $x_0$ yields in a similar manner the $k \times n$ matrix,

$$\int_0^T \mathfrak{B}' X(t)' X(t)\, dt,$$

which is the block of elements in the lower left corner of $\mathfrak{M}$.

Differentiating the left member of (8.15) with respect to $x_0$ yields in a certain $k \times k$ matrix $\mathfrak{K}$, whose properties need no further investigation and which furnishes the block of elements in the lower right corner of $\mathfrak{M}$.

In other words, the jacobian we are interested in turns out to be $\det \mathfrak{M}$, which we know from Lemma 7.2 is not zero. This establishes Theorem 8.1, except for a few details which we leave to the reader.

THEOREM 8.2. *If the $k$-vector $c$ and the scalar $\mu$ satisfy the $k$-vector equation,*

$$\alpha(c, \mu) = 0, \tag{8.16}$$

*then the $n$-vector function $x(t, c, \mu)$ is an autosynartetic solution of* (6.1) *under the transformation* (6.2). *Here it is understood that $\alpha(c, \mu)$ and $x(t, c, \mu)$ are as introduced in Theorem* 8.1.

*Proof.* Because of (8.2) and (8.16), $x(t, c, \mu)$ is a solution of (6.1). Because of (8.3) and Theorem 3.4 (with $t_0 = 0$, etc.), $x(t, c, \mu)$ is also autosynartetic.

THEOREM 8.3. *If $\bar{x}(t)$ is an autosynartetic solution of* (6.1) *under* (6.2) *and if $|\mu|$ and $|x(t) - \phi(t)|$ are sufficiently small, there exists a $k$-vector $c$ such that* (8.16) *is satisfied and such that $\bar{x}(t) \equiv x(t, c, \mu)$.*

*Proof.* Using the function $\psi(t, x_0, \alpha, \mu)$ introduced at the beginning of the proof of Theorem 8.1 (cf. (8.6)), we see from the uniqueness theorem for differential equations that

$$\bar{x}(t) = \psi(t, \bar{x}(0), 0, \mu). \tag{8.17}$$

Since $\bar{x}(t)$ is given as autosynartetic under (6.2), Theorem 3.4 shows that $\bar{x}[P(0, \bar{x}(0), \mu)] = h(0, \bar{x}(0), \mu)$, or, using (8.17), we get

$$\psi[P(0, \bar{x}(0), \mu), \bar{x}(0), 0, \mu] = h(0, \bar{x}(0), \mu). \tag{8.18}$$

Since $\bar{x}(0) = \psi(0, \bar{x}(0), 0, \mu)$, we see from (8.18) that (8.14) is satisfied by $x_0 = \bar{x}(0)$ and $\alpha = 0$.

We define the $k$-vector $c$ by means of the system of linear equations

$$(8.19) \qquad \int_0^T \mathfrak{B}'X(t)'[\psi(t,\bar{x}(0),0,\mu)-\phi(t)-X(t)\mathfrak{B}c]dt=0.$$

The determinant of this system in the components of $c$ is seen by (7.8) to be different from zero. Hence (8.19) gives a valid definition of $c$; and moreover $|c|$ will be small if $|\mu|$ and $|\bar{x}(0)-\phi(0)|$ are sufficiently small. With this value of $c$ we see, by comparing (8.19) with (8.15), that the latter equation is satisfied by $x_0=\bar{x}(0)$ and $\alpha=0$.

But the complete implicit function theorem contains a statement to the effect that there are no solutions near the given solution except those furnished by the implicitly defined functions. Applying this statement to the system (8.14)-(8.15), and using the notation $x_0(c,\mu)$, as well as $\alpha(c,\mu)$, introduced in the proof of Theorem 8.1, we find that $\bar{x}(0)=x_0(c,\mu)$ and $0=\alpha(c,\mu)$. Hence by (8.17) we have

$$x(t,c,\mu)=\psi[t,x_0(c,\mu),\alpha(c,\mu),\mu]=\psi[t,\bar{x}(0),0,\mu]=\bar{x}(t),$$

as we wished to prove.

DEFINITION 8.1. *The $k$-vector function $\alpha(c,\mu)$ being introduced as in Theorem* 8.1, *the $k$-vector equation* (8.16) (*or the system of $k$ scalar equations* (8.16)) *will be called the bifurcation equation* (*or equations*) *for the autosynartetic solution $\phi(t)$ of degeneracy $k$.*

Theorems 8.2 and 8.3 may be summarized by the statement that the problem of the perturbation of an autosynartetic solution of degeneracy $k$ can always be reduced to the problem of solving a system of $k$ "bifurcation" equations instead of the generally larger system of $n$ equations like (6.4). An example of the advantage of using bifurcation equations is indicated in the following

THEOREM 8.4. *Suppose that the system* (6.1) *admits $l$ independent autosynartetic first integrals of class $C'$ in $t$, $x$, and $\mu$. Suppose also that for $\mu=0$, $\phi(t)$ is a given autosynartetic solution of* (6.1) *of degeneracy $k\geqq l$; and suppose that the $k$-vector function $\alpha(c,\mu)$ is set up as in Theorem* 8.1. *Then there exists an $l\times k$ matrix function $Q(c,\mu)$ of rank $l$* (*when $|c|$ and $|\mu|$ are sufficiently small*), *such that*

$$(8.20) \qquad Q(c,\mu)\alpha(c,\mu)\equiv 0.$$

*Proof.* Let the $l$ autosynartetic first integrals be represented as the components of the $l$-vector $\Lambda(t,x,\mu)$. From Theorem 5.2, we know that the rows of the $l\times n$ matrix $\Lambda_x(t,x,\mu)$ evaluated for $x=\phi(t)$ and $\mu=0$ are

autosynartetic solutions of the adjoint to the variational equations. Since also the $k \times n$ matrix $\Xi(t)$ of (7.7) is such that its $k$ rows afford a full set of linearly independent autosynartetic solutions of the adjoint system it is clear that the rows of $\Lambda_x(t, \phi(t), 0)$ are linear combinations of the rows of $\Xi(t)$. In other words there exists a constant $l \times k$ matrix $D$ such that

$$\Lambda_x(t, \phi(t), 0) = D\Xi(t). \tag{8.21}$$

Moreover, since the $l$ first integrals are independent $\Lambda_x(t, \phi(t), 0)$, and hence $D$, must be of rank $l$. If now we define the $l \times n$ matrix

$$\zeta(t, c, \mu) = \Lambda_x(t, x(t, c, \mu), \mu) - D\Xi(t), \tag{8.22}$$

where $x(t, c, \mu)$ has the meaning explained in Theorem 8.1, we see from the continuity of $x(t, c, \mu)$ and of $\Lambda_x(t, x, \mu)$ and from (8.21) that $\zeta(t, c, \mu) \to 0$ uniformly on any finite $t$-interval as $|c|$ and $|\mu| \to 0$.

Since $\Lambda(t, x, \mu)$ is a vector first integral, we have

$$\Lambda_t(t, x, \mu) + \Lambda_x(t, x, \mu) f(t, x, \mu) \equiv 0.$$

In this identity, we replace $x$ by the vector $x(t, c, \mu)$ of Theorem 8.1 and integrate from 0 to

$$T(c, \mu) = P(0, x(0, c, \mu), \mu). \tag{8.23}$$

We thus obtain

$$\begin{aligned} &\int_0^{T(c,\mu)} \Lambda_x(t, x(t, c, \mu), \mu) f(t, x(t, c, \mu), \mu)\, dt \\ &\qquad + \int_0^{T(c,\mu)} \Lambda_t(t, x(t, c, \mu), \mu)\, dt \equiv 0. \end{aligned} \tag{8.24}$$

Since $\Lambda$ is autosynartetic, we have (by Definition 5.1)

$$\Lambda(t, x, \mu) \equiv \Lambda(P(t, x, \mu), h(t, x, \mu), \mu). \tag{8.25}$$

Setting $t = 0$, $x = x(0, c, \mu)$, and remembering that $h(0, x(0, c, \mu), \mu) = x(T(c, \mu), c, \mu)$ by Theorem 8.1 and (8.23), we find from (8.25) that $\Lambda[T(c, \mu), x(T(c, \mu), c, \mu), \mu] - \Lambda(0, x(0, c, \mu), \mu) \equiv 0$. Hence

$$\int_0^{T(c,\mu)} (d/dt)[\Lambda[t, x(t \cdot c, \mu), \mu]]\, dt \equiv 0.$$

Hence

$$\text{(8.26)}\qquad \int_0^{T(c,\mu)} \Lambda_x[t, x(t,c,\mu),\mu] x_t(t,c,\mu)\,dt + \int_0^{T(c,\mu)} \Lambda_t[t, x(t,c,\mu),\mu]\,dt \equiv 0.$$

Substracting (8.26) from (8.24) yields

$$\int_0^{T(c,\mu)} \Lambda_x[t, x(t,c,\mu),\mu][f(t, x(t,c,\mu),\mu) - x_t(t,c,\mu)]\,dt \equiv 0.$$

But, from (8.2), this last identity may be written

$$\int_0^{T(c,\mu)} \Lambda_x[t, x(t,c,\mu),\mu]\Xi(t)'\alpha(c,\mu)\,dt \equiv 0.$$

We therefore obtain (8.20), if we let

$$Q(c,\mu) = \int_0^{T(c,\mu)} \Lambda_x(t, x(t,c,\mu),\mu)\Xi(t)'dt.$$

Evidently from (8.22)

$$Q(c,\mu) = D\int_0^T \Xi(t)\Xi(t)'dt + D\int_T^{T(c,\mu)} \Xi(t)\Xi(t)'dt + \int_0^{T(c,\mu)} \zeta(t,c,\mu)\Xi(t)'dt.$$

Now $T(c,\mu) \to T(0,0) = T$ as $c \to 0$ and $\mu \to 0$ and $\zeta(t,c,\mu) \to 0$ uniformly. We also know that $\int_0^T \Xi(t)\Xi(t)'dt$ is a non-singular $k \times k$ matrix (cf. Lemma 7.1 (V)). It therefore follows that $Q(c,\mu)$ must, like $D$, have the rank $l$ for $|c|$ and $|\mu|$ sufficiently small.

A simple corollary of Theorem 8.4 is the following

THEOREM 8.5. *If there are $l$ independent autosynartetic first integrals, the perturbation of any autosynartetic solution of degeneracy $k$ (necessarily $\geqq l$ by Theorem 5.3) may be effected by the solution of $k - l$ of the bifurcation equations, the other $l$ bifurcation equations being then automatically satisfied. In particular, if $k = l$, the bifurcation equations are all identically satisfied.*

*Proof.* Since $Q(c,\mu)$ is of rank $l$, we may solve (8.20) for $l$ of the

components of $\alpha$, say, $\alpha_1, \cdots, \alpha_l$ in terms of the other components $\alpha_{l+1}, \cdots, \alpha_k$ so that we have equations of the form

$$\alpha_i(c,\mu) = \sum_{j=l+1}^{k} q_{ij}(c,\mu)\alpha_j(c,\mu), \qquad i = 1, \cdots, l.$$

Hence, if $\alpha_j(c,\mu) = 0$ for $j = l+1, \cdots, k$, we also have $\alpha_i(c,\mu) = 0$ for $i = 1, \cdots, l$.

THEOREM 8.6. *Suppose that the system $dx/dt = f(t,x,0)$ admits $l$ independent autosynartetic first integrals of class $C'$ in $t$ and $x$ (even though the system $dx/dt = f(t,x,\mu)$ for $\mu \neq 0$ may not). Suppose also that $\phi(t)$ is a given autosynartetic solution of (6.1) under (6.2) when $\mu = 0$ and that the $k$-vector function $\alpha(c,\mu)$ is set up as in Theorem 8.1. Then there exists an $l \times k$ matrix function $Q(c)$ of rank $l$, when $|c|$ is sufficiently small, such that*

$$Q(c)\alpha(c,0) \equiv 0. \tag{8.27}$$

*Proof.* Consider the modified system $dx/dt = f^*(t,x,\mu)$ where $f^*(t,x,\mu) \equiv f(t,x,0)$ is a constant with respect to variation in $\mu$. Then the given $l$ autosynartetic first integrals of $\dot{x} = f(t,x,0)$, which are also independent of $\mu$, are autosynartetic first integrals of $dx/dt = f^*(t,x,\mu)$ under the modified transformation $s = P^*(t,x,\mu) \equiv P(t,x,0)$, $y = h^*(t,x,\mu) \equiv h(t,x,0)$. Hence Theorem 8.4 applies to the modified system under the modified transformation and we get an $l \times k$ matrix, $Q^*(c,\mu)$, of rank $l$ for $|\mu|$ and $|c|$ sufficiently small, such that

$$Q^*(c,\mu)\alpha^*(c,\mu) \equiv 0, \tag{8.28}$$

where $\alpha^*(c,\mu)$ is to the modified system as $\alpha(c,\mu)$ is to the original system. The fact that $\alpha^*(c,\mu) = \alpha(c,0)$ follows from the following two facts: (I) The equations for the determination of $\alpha^*(c,\mu)$ and $x^*_0(c,\mu)$, namely equations (8.14) and (8.15) set up for the modified system, are seen to be completely independent of $\mu$, so that $\alpha^*(c,\mu) = \alpha^*(c,0)$. (II) These same equations are identical with the original equations (8.14) and (8.15) when $\mu = 0$, so that $\alpha^*(c,0) = \alpha(c,0)$. Thus, from 8.28), we find that $Q^*(c,\mu)\alpha(c,0) \equiv 0$, from which we get (8.27) by choosing $Q(c) = Q^*(c,0)$.

We next turn to the consideration of properties of the bifurcation equations not dependent on the existence of autosynartetic first integrals. The *general* situation is exhaustively treated in the next two theorems.

THEOREM 8.7. *The left hand member of the bifurcation equation* (8.16) *always possesses the following two properties:*

$$\alpha(0,0)=0 \tag{8.29}$$

$$\alpha_c(0,0)=0. \tag{8.30}$$

*Proof.* From (8.5) and (8.2) we have

$$d\phi(t)/dt=f[t,\phi(t),0]-\Xi(t)'\alpha(0,0).$$

But $\phi(t)$ was given at the outset as a solution of (6.1) when $\mu=0$. Hence

$$d\phi(t)/dt=f[t,\phi(t),0].$$

Therefore $\Xi(t)'\alpha(0,0)=0$. From (7.7) we know that the rank of $\Xi$ is $k$. Hence $\alpha(0,0)=0$.

Incidentally, this simple proof of (8.29) or (8.13) as a necessary consequence of (8.2) and (8.5) is one of the details left to the reader at the end of the proof of Theorem 8.1. It is a step in the proof of the *uniqueness* part of Theorem 8.1.

We next establish (8.30) as follows: With the help of (8.8) we write equation (8.14) in the form,

$$\psi[P(0,x_0,\mu),x_0,\alpha,\mu]-h(0,x_0,\mu)=0.$$

This equation is satisfied by $x_0=x_0(c,\mu)$ and $\alpha=\alpha(c,\mu)$. Hence, upon setting $\mu=0$, we obtain

$$\psi[P(0,x_0(c,0),0),x_0(c,0),\alpha(c,0),0]-h(0,x_0(c,0),0)=0.$$

Differentiating with respect to $c$, freely using the various properties of $\psi$ indicated in formulas (8.6) to (8.11), and also remembering the definition of $B(0)$ given by (4.9 alt.), we find, on setting $c=0$ and $x_0(0,0)=\phi(0)$, that

$$-B(0)(\partial x_0/\partial c)+X[P(0,\phi(0),0)](\partial x_0/\partial c)$$
$$+\sigma[P(0,\phi(0),0)](\partial\alpha(0,0)/\partial c)=0.$$

Since $B(0)=B$, $X(0)=I$, and $P(0,\phi(0),0)=T$, we therefore have

$$[X(T)-BX(0)](\partial x_0/\partial c)+\sigma(T)(\partial\alpha(0,0)/\partial c)=0.$$

Hence, from (7.6), $\mathfrak{A}\sigma(T)(\partial\alpha(0,0)/\partial c) = 0$, while from (8.11), (7.7), and (V) of Lemma 7.1, we see that

$$\mathfrak{A}\sigma(T) = -\mathfrak{A}\int_0^T X(T)X(s)^{-1}\Xi(s)'ds = -\int_0^T \Xi(s)\Xi(s)'ds$$

is a non-singular matrix, so that (8.30) follows immediately.

THEOREM 8.8. *The bifurcation equations in general have no distinctive properties other than those specified in Theorem 8.7. More precisely:*

*Let $\beta(c,\mu)$ be an arbitrary $k$-vector function of the $k$-vector $c$ and the scalar $\mu$. Let $\beta(c,\mu)$ be of class $C'$, and let it vanish together with its first partial derivatives with respect to the components of $c$ when $c = 0$ and $\mu = 0$. Let $n$ be any integer $\geqq k$ and suppose $T$ is any positive number. Then there exists a system (6.1) of order $n$ such that $f(t+T,x,\mu) \equiv f(t,x,\mu)$, which possesses, for $\mu = 0$, a periodic solution of period $T$ and degeneracy $k$, and whose bifurcation equations are $\beta(c,\mu) = 0$.*

*Proof.* We proceed to set up such a system. For this purpose, the first $k$ components of the $n$-vector $x$ will be thought of as a $k$-vector $x'$, while the last $(n-k)$ components of $x$ will be called the $(n-k)$-vector $x''$. Let $A''(t)$ be any $(n-k)\times(n-k)$ matrix of continuous periodic functions of $t$ of period, $T$, such that the linear system $dx''/dt = A''(t)x''$ has no non-trivial periodic solution. $A''(t)$ could be a constant matrix, if desired. Then it is claimed that the system

$$(8.31) \qquad dx'/dt = \beta(x',\mu), \qquad dx''/dt = A''(t)x'',$$

admits, for $\mu = 0$, the periodic solution $x' = 0$, $x'' = 0$, and that the bifurcation equations (8.16) for the perturbation of this solution are such that

$$(8.32) \qquad \alpha(c,\mu) \equiv \beta(c,\mu),$$

where, of course, it is understood that $\alpha(c,\mu)$ is to constructed so as to satisfy the conditions of Theorem 8.1. Since $\beta_{x'}(0,0) = 0$, the variational equations take the form

$$d\xi'/dt = 0, \qquad d\xi''/dt = A''(t)\xi''.$$

The $n \times n$ matrix solution $X(t)$ of this system such that $X(0) = I$ is seen to have the form

$$X(t)=\begin{pmatrix} I' & 0 \\ 0 & X''(t)\end{pmatrix},$$

where $I'$ is the $k\times k$ identity matrix while $X''(t)$ is an $(n-k)\times(n-k)$ matrix, no linear combination of whose columns is periodic, while $X''(0)=I''$, the $(n-k)\times(n-k)$ identity matrix. In formula (7.6) the matrices $B$, $\mathfrak{A}$, and $\mathfrak{B}$ should now evidently be interpreted as follows:

$$B=I, \qquad \mathfrak{A}=(I',0), \qquad \mathfrak{B}=\begin{pmatrix} I' \\ 0\end{pmatrix}.$$

From (7.7) we then find that $\Xi(s)=(I',0)$ so that $\Xi(t)'\alpha=\begin{pmatrix} I'\alpha \\ 0\end{pmatrix}=\begin{pmatrix} \alpha \\ 0\end{pmatrix}$. Hence (8.2) becomes

$$\text{(I)}\qquad \begin{aligned} x'_t(t,c,\mu) &= \beta(x'(t,c,\mu),\mu)-\alpha \\ x''_t(t,c,\mu) &= A''(t)x''(t,c,\mu). \end{aligned}$$

The condition (8.3) reduces, of course, to

$$\text{(II)}\qquad x'(T,c,\mu)=x'(0,c,\mu), \qquad x''(T,c,\mu)=x''(0,c,\mu).$$

From the fact that $\phi(t)\equiv 0$ in the present example, a routine calculation shows that (8.4) reduces to

$$\text{(III)}\qquad \int_0^T (x'(t,c,\mu)-c)\,dt=0,$$

while (8.5) is simply

$$\text{(IV)}\qquad x(t,,0,0)=0.$$

According to Theorem 8.1, these conditions determine $x(t,c,\mu)$ and $\alpha(c,\mu)$ *uniquely*. But we see by inspection that all conditions are satisfied, if we take $x'(t,c,\mu)=c$, $x''(t,c,\mu)=0$, and $\alpha=\alpha(c,\mu)=\beta(c,\mu)$. This completes the proof of (8.32) and, therefore, of the theorem.

We close this section with a brief consideration of the case when (6.1) is transferred into itself by a family of transformations,

$$(8.33)\qquad s=P(t,x,\mu,\lambda), \qquad y=h(t,x,\mu,\lambda),$$

where the parameter $\lambda$ is a scalar, or, more generally an $m$-vector. The base solution $\phi(t)$ for $\mu=0$ is supposed to be autosynartetic under (8.33) for $\mu=0$ and $\lambda=0$. We proceed exactly as in the earlier case when the transformation did not depend on $\lambda$; only now our bifurcation equation (as well

as the function $x(t, c, \mu)$ of Theorem 8.1) will depend on $\lambda$. We therefore write our bifurcation equation in the form,

$$\alpha(c, \mu, \lambda) = 0. \tag{8.34}$$

Evidently the presence of the new variable $\lambda$ increases the probability of our being able to solve the bifurcation equations. Thus (8.34) may have no solution for which $\mu \neq 0$ and $\lambda = 0$, but it might very well have some for which $\mu \neq 0$ and $\lambda \neq 0$. In the latter case, we are led to an autosynartetic solution of (6.1) under (8.33) for values of $\lambda \neq 0$.

The most familiar case in which this situation arises is in the perturbation of a non-constant periodic solution of an autonomous system. Some remarks about how this case fits into our general theory have already appeared in Sections 1 and 5.

**9. Miscellaneous comments.** The reader may notice that the author has abandoned the method of integral equations used in his previous papers, especially [3]. The reason for this is primarily that the interval of integration, which would be between 0 and $P(0, x(0, \mu), \mu)$ is, in general, dependent on the unknown solution $x(t, \mu)$ as well as upon the parameter $\mu$. A secondary reason is that the autosynartetic boundary conditions for the variational equations are not the same (in general) for different autosynartetic solutions. In the special case, when $P(t, x, \mu)$ is independent of both $x$ and $\mu$ and $h(t, x)$ is linear and homogeneous in $x$, as is the case in the study of the perturbation of nonautonomous periodic solutions, these objections are no longer valid. In such cases the integral equation method is entirely feasible and, indeed, would afford the best available method of estimating the $\mu$-interval over which perturbation is possible.

Likewise the reader will observe that only a part of Lemma 7.1 is essential for the purposes of the present paper. In particular no use is made of the "generalized Green's matrix," $G(t, s)$. Nevertheless, for the sake of completeness and in view of the indispensability of this $G(t, s)$ for the integral equation method in the fairly general case mentioned above when it is feasible and useful to employ this technique, it seemed desirable to take the opportunity of presenting Lemma 7.1 in its complete form.

Our theory of the bifurcation equation, when specialized to the periodic case, is seen to parallel very closely a theory outlined by Friedrichs [1]. Friedrichs, however, would seem to replace our condition (8.4) by a condition of the form,

$$\omega x(0, c, \mu) = c, \tag{9.1}$$

where $\omega$ is a constant $k \times n$-matrix, while $c$ is a $k$-vector, as before. The present author felt that (8.4) was a more "natural" condition, perhaps because of his previous exposure to the work of Ernst Hölder [2]. It is probably possible to include both conditions, namely (8.4) and (9.1), under a more general condition, if we tamper somewhat with our Lemma 7.1, using a more generalized notion of orthogonality, which would involve the use of weight factors and Stieltjes integration. Such a generalization would probably be a mere tour de force devoid of substantially new results.

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## CALCULATION OF CLASS NUMBERS BY DECOMPOSITION INTO THREE INTEGRAL SQUARES IN THE FIELDS OF $2^{\frac{1}{2}}$ AND $3^{\frac{1}{2}}$.*[1]

By Harvey Cohn.[2]

**12. Introduction.** The results presented here are a continuation of earlier results on modular functions of two complex variables of the author [16] relating to the field of $2^{\frac{1}{2}}$ and $3^{\frac{1}{2}}$. Their significance is that these results broaden the range of those modular functions which illustrate elegant properties too refined to be concluded wholly from such vast theoretical structures as that of Siegel [15]. The results have, in addition, a useful relationship to class-number calculations, which we proceed to summarize.

Consider the real quadratic field $R(D^{\frac{1}{2}})$ with discriminant $D > 0$, and, within the field, consider a totally positive integer $\mu(\gg 0)$ which might be specialized to be rational. (When $D = 1$, the field will be taken as rational.) We assume $\mu$ to be square-free for simplicity. We let $H(D^{\frac{1}{2}}, (-\mu)^{\frac{1}{2}})$ denote the class number of $R(D^{\frac{1}{2}}, (-\mu)^{\frac{1}{2}})$ and we let $A_3(\nu)$ denote the number (possibly zero) of decompositions of type

$$\nu = \xi_1^2 + \xi_2^2 + \xi_3^2, \tag{12.1}$$

where $\xi_i$ are integers in $R(D^{\frac{1}{2}})$ and every permutation or change of sign in the triple $(\xi_1, \xi_2, \xi_3)$ is tallied as an additional decomposition.

The class-number relationships which concern us, have essentially the form

$$\begin{cases} A_3(\mu\gamma^2) = G \cdot H(D^{\frac{1}{2}}, (-\mu)^{\frac{1}{2}}) \\ \mu \text{ square-free and totally positive } (\gg 0), \end{cases} \tag{12.2}$$

where $D = 1, 5, 8,$ and $12$. The factor $\gamma^2$ is needed in (12.2) because when $D = 8$ or $12$, $\nu$ must belong to $\mathfrak{O}_2$ the ring of integers $\nu$ of $R(D^{\frac{1}{2}})$ of form $a + bD^{\frac{1}{2}}$. Thus even if $\mu \notin \mathfrak{O}_2$, we can conclude $\mu\gamma^2 \in \mathfrak{O}_2$ when we set

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[2] The numbering of sections and bibliographical items is consecutive with the earlier paper [16]. With the exception of some general theorems of § 7 (et al.) this paper has self-contained proofs, but perpetuates the earlier notation.

$$(12.3)\qquad \begin{cases} \gamma = 1 & \text{for } D = 1, 5, \\ \gamma = 1 & \text{for } D = 8, 12 \text{ and } \mu \in \mathfrak{O}_2, \\ \gamma = 2^{\frac{1}{2}} & \text{for } D = 8 \text{ and } \mu \notin \mathfrak{O}_2, \\ \gamma = 1 + 3^{\frac{1}{2}} & \text{for } D = 12 \text{ and } \mu \notin \mathfrak{O}_2. \end{cases}$$

The values of $G$ are given in the accompanying table. These values depend on the manner in which the ideal (2) factors in $R(D^{\frac{1}{2}}, (-\mu)^{\frac{1}{2}})$ into distinct prime ideal factors $\mathfrak{p}$, $\mathfrak{q}$. A list of cases based on easily recognized residue classes is given in the adjoining column:

| $D$ | $G$ | Factors of (2) | Residue classes of $\mu$ | |
|---|---|---|---|---|
| 1 | 0 | $\mathfrak{pq}$ | 7 | (mod 8) |
| (Gauss) | 24 | $\mathfrak{p}$ | 3 | " |
| | 12 | $\mathfrak{p}^2$ | misc. | " |
| 5 | 96 | $\mathfrak{pq}$ | $7, 3, (13 \pm 5^{\frac{1}{2}})/2, (9 \pm 3 \cdot 5^{\frac{1}{2}})/2$ | (mod 8) |
| (Maass) | 120 | $\mathfrak{p}$ | $5 \pm 2 \cdot 5^{\frac{1}{2}}, (5 \pm 5^{\frac{1}{2}})/2, (9 \pm 5 \cdot 5^{\frac{1}{2}})/2$ | " |
| | 12 | $\mathfrak{p}^2$ | misc. | |
| 8 | 48 | $\mathfrak{p}^2\mathfrak{q}^2$ | $7, 5 + 2 \cdot 2^{\frac{1}{2}}$ | (mod $4 \cdot 2^{\frac{1}{2}}$) |
| | 96 | $\mathfrak{p}^2$ | $3, 1 + 2 \cdot 2^{\frac{1}{2}}$ | " |
| | 24 | $\mathfrak{p}^4$ | misc. | " |
| 12 | 24* | $\mathfrak{p}^2\mathfrak{q}^2$ | 7, 5 | (mod $4(1 + 3^{\frac{1}{2}})$) |
| | 48* | $\mathfrak{p}^2$ | 3, 1 | " |
| | 12* | $\mathfrak{p}^4$ | misc. | " |

(* See further explanation below.)

Here, for simplicity, we have further restricted $\mu$ to the case where the fields $R(D^{\frac{1}{2}})$ and $R(D^{\frac{1}{2}}, (-\mu)^{\frac{1}{2}})$ have precisely the same units. The exceptions are as follows (ignoring factors of $\mu$ which are squares for $R(D^{\frac{1}{2}})$):

$$(12.4)\qquad \begin{cases} \mu = 1, 3 & \text{for } D = 1, 5, 8, 12, \\ \mu = (5 + 5^{\frac{1}{2}})/2 & \text{for } D = 5, \\ \mu = 2 + 3^{\frac{1}{2}} & \text{for } D = 12. \end{cases}$$

In the first three cases, a complex root of unity is introduced into $R(D^{\frac{1}{2}}, (-\mu)^{\frac{1}{2}})$ and in the last case a new fundamental unit $(-2 - 3^{\frac{1}{2}})^{\frac{1}{2}}$ is introduced that was not present in $R(D^{\frac{1}{2}})$. An independent calculation reveals that in the cases (12.4) the class number $H(D^{\frac{1}{2}}, (-\mu)^{\frac{1}{2}})$ is unity, when $D = 1, 5, 8$. (A formula of type (20.16) below, reconciles these exceptions.)

The basic formula (12.2) fails, however, when $D = 12$ (see * in the table), and the formula (12.2) is an "approximation" valid only to the extent described in the next section. Considering $D = 8$, we know *for any integers a, b with* $a > |b| 8^{\frac{1}{2}}$, *we can represent*

$$(12.5) \qquad a + 2b2^{\frac{1}{2}} = (x_1 + y_1 2^{\frac{1}{2}})^2 + (x_2 + y_2 2^{\frac{1}{2}})^2 + (x_3 + y_3 2^{\frac{1}{2}})^2$$

*in terms of integers* $x_i$, $y_i$. Thus a theorem for three squares supersedes the one for four squares in the earlier work [16]. The preponderance of numerical data [3] makes it impossible, however, to reject the hypothesis that three squares suffice for $D = 12$.

**13. Remarks on method.** There are three different methods for establishing a formula of type (12.2). The first method is the method of Gauss which is based on a direct connection between the enumeration of quadratic forms and representations like (12.1). This method as been applied with success only to the rational case, $D = 1$.

The other two methods involve the use of some kind of theta series $\Theta(\tau)$. The function $\Theta^3(\tau)$ is, by some means or other, expanded into a "singular series" $\Psi(\tau)$ to yield identity (12.3) through a comparison of coefficients, as did Jacobi in his classic proof for $\Theta^4(\tau)$ in the rational field.

There are differences, however, in general methods for establishing $\Theta^3 = \Psi$, the basic identity. The "direct" method was applied (to the $\Theta^3$ case) by Kronecker [4, p. 109], (see Mordell's version [23]), but in the rational case only, ($D = 1$).

We shall see that, in the quadratic case, $\Theta^3$ and $\Psi$ satisfy a system of linear functional equations (like that in § 16 below), which happens to make for a one-parameter family of solutions. This method could have been used for $D = 5$ by Maass; we shall use it for $D = 8$ (as done earlier [16] for $\Theta^4$). Indeed, the identities in § 24 are almost identical to those that would emerge for $D = 5$, so, in effect, an alternate proof of Maass' result [12] is provided by this paper.

The less direct but deeper method is due to Siegel [15] and is based on the fact that the form $\xi_1^2 + \xi_2^2 + \xi_3^2$ in $R(D^{\frac{1}{2}})$ is the only equivalence class in its genus. The method was carried out by Maass for $D = 1$ [21], and $D = 5$ [12]; (the discovery of the role of the $\Theta$-function and class number in modular functions is due to Hecke [17]).

It should be remarked that, however straightforward it may be, the *calculation* of the singular series §§ 14-21 is a formidable matter, although the "proof proper" consists of §§ 22-24, where it is proved that $\Theta^3 - \Psi = 0$ for $D = 8$.

We shall see that when $D = 12$, $\Theta^3 - \Psi$ is a cusp-form not identically zero as evidenced by the failure of equation (12.2), or more specifically by the failure at its source (20.16), below. The case $D = 12$, however, as the case of four squares, appears to be within the scope of some kind of *exact* formula for (say) $A_3(\mu\gamma^2)$. For example, preliminary calculations show formula (12.2) to be correct when $D = 12$, if $\mu$ is a square-free rational integer $> 1$ and relatively prime to 6.

A continuing study for three and four square representations when $D = 12$ is being undertaken with exact formulas as a goal.

## Calculation of Singular Series.

**14. Notation.** We continue the earlier notation with Greek letters denoting algebraic integers, Roman letters denoting rational integers (except for special function-theoretic symbols), and the generalized Norm $N$ and Trace $S$ applicable to conjugates in $R(D^{\frac{1}{2}})$ and our two so-called "conjugate" continuous variables $\tau$ and $\tau'$, (each complex).

We consider the two fields with given discriminant and generator:

$$(14.1) \qquad \begin{cases} D = 8, & \eta = 2^{\frac{1}{2}}, \\ D = 12, & \eta = 1 + 3^{\frac{1}{2}}. \end{cases}$$

They share the properties of being Euclidean, having the fundamental unit

$$(14.2) \qquad \epsilon = 1 + \eta$$

and having the ring of all algebraic integers equal to

$$(14.3) \qquad \mathfrak{O} = [1, \epsilon] = [1, \eta],$$

(which is required in setting up generators of the modular group, as in § 2). In either case,

$$(14.4) \qquad (\eta^2) = (2), \qquad N(\eta) = -2,$$

$$(14.5) \qquad \mathfrak{O}_2 = [1, 2\eta] = [1, D^{\frac{1}{2}}].$$

We introduce the simplifying symbol

$$(14.6) \qquad \begin{aligned} e(\zeta) &= \exp \pi i S(\zeta/D^{\frac{1}{2}}) \\ &= \exp \pi i (\zeta - \zeta')/D^{\frac{1}{2}}. \end{aligned}$$

In particular,

$$(14.7) \qquad e([r + s\eta]/t) = \exp \pi i s/t;$$

thus for the algebraic integer $\rho$ (in $\mathfrak{O}$)

$$e(\rho) = 1 \text{ if and only if } \rho \in \mathfrak{O}_2. \tag{14.8}$$

Finally, using the complex variables in the argument of $e(\rho)$, we redefine the earlier $\Theta_{d,c}(\tau)$ as follows:

$$\Theta(d, c; \tau) = \sum_{\nu} e(\nu d\eta + [\nu + c\eta/2]^2\tau) \tag{14.9}$$

summed over all $\nu$ in $\mathfrak{O}$, where,

$$\begin{cases} \operatorname{Im} \tau > 0, \\ \operatorname{Im} \tau' < 0. \end{cases} \tag{14.10}$$

The following transformation laws are determined from formulas:

$$\Theta(d+2, c; \tau) = \Theta(d, c+2; \tau) = \Theta(d, c; \tau), \tag{14.11}$$

$$\Theta(d, c; \epsilon^2\tau) = \Theta(d, c; \tau), \tag{14.12}$$

$$\Theta(d, c; \tau + \eta) = \Theta(d+1, c; \tau) e(c^2\eta^3/4), \tag{14.13}$$

$$\Theta(d, c; \tau + 1) = \Theta(d + c, c; \tau) e(c^2\eta^2/4), \tag{14.14}$$

$$\Theta(d, c; -1/\tau) = e(cd\eta^2/2)\Theta(c, d; \tau) N(\tau)^{\frac{1}{2}}, \tag{14.15}$$

for the principal square root of $N(\tau)$.

**15. Asymptotic relations.** To calculate the singular series, we consider $\Theta(d, c; \tau)$ near $\tau = \alpha/\beta$, $(\tau' = \alpha'/\beta')$, an irreducible, algebraic rational in $R(D^{\frac{1}{2}})$.

$$\Theta(d, c; \alpha/\beta + \lambda) = \sum_{\nu} e(\nu d\eta + [\nu + c\eta/2]^2\alpha/\beta) e([\nu + c\eta/2]^2\lambda), \tag{15.1}$$

where $\lambda$, $\lambda'$ are a new pair of complex variables satisfying condition (14.10). Then, writing

$$\nu = \nu_1 + 2\beta\nu_2, \ (\nu_1 \bmod 2\beta, \nu_2 \text{ arbitrary}), \tag{15.2}$$

we find

$$\begin{aligned} &\Theta(d, c; \alpha/\beta + \lambda) \\ &\quad = \sum_{\nu_1} e(\nu_1 d\eta + [\nu_1 + c\eta/2]^2\alpha/\beta) \sum_{\nu_2} e([\nu_1 + 2\beta\nu_2 + c\eta/2]^2\lambda). \end{aligned} \tag{15.3}$$

But the inner sum, according to the general method, is asymptotically independent of $\nu_1$ and $c$; thus

$$\sum_{\nu_2} \approx \Theta(0, 0, 4\beta^2\lambda) = \Theta(0, 0, -1/(4\beta^2\lambda))/N(4\beta^2\lambda)^{\frac{1}{2}} \tag{15.4}$$

according to formula (14.9). Finally,

$$\Theta(d, c; \alpha/\beta + \lambda) \approx G_{d,c}(\alpha/\beta)/4\,|\,N(\beta)\,|\,N(\lambda)^{\frac{1}{2}}, \tag{15.5}$$

where

$$G_{d,c}(\alpha/\beta) = \sum_{\nu \bmod 2\beta} e(\nu d\eta + (\nu + c\eta/2)^2\alpha/\beta). \tag{15.6}$$

At this point, we examine the Gaussian-type sum $G_{d,c}(\alpha/\beta)$ more critically and note a new type

$$H_{d,c}(\alpha/\beta) = \sum_{\nu \bmod \beta} e(\nu d\eta + [\nu + c\eta/2]^2\alpha/\beta), \tag{15.7}$$

when

$$(\alpha + d\eta)(\beta + c\eta) \in \mathfrak{D}_2. \tag{15.8}$$

Such a sum is characteristic of the cases $D = 8$, 12. In the work of Maass [22, p. 716], no condition like (15.8) arises. Indeed, unless (15.8) is valid, the residues (mod $2\beta$) $\nu$ and $\nu + \beta$ make cancelling contributions to the sum (15.7). Thus, calling $H_{d,c}(\alpha/\beta) = 0$ when (15.8) fails, we have

$$G_{d,c}(\alpha/\beta) = 4H_{d,c}(\alpha/\beta). \tag{15.9}$$

Finally, changing variables, we say, for $\tau$, $\tau'$ near $\alpha/\beta$, $\alpha'/\beta'$,

$$\Theta(d, c; \tau) \approx \begin{cases} H_{d,c}(\alpha/\beta)\operatorname{sgn} N(\beta)/N(\beta)^{\frac{1}{2}}N(\beta\tau - \alpha)^{\frac{1}{2}}, \text{ (if } \alpha/\beta \neq 1/0); \\ \delta_{0,c}, \text{ (if } \alpha/\beta = 1/0). \end{cases} \tag{15.10}$$

(The last item was inserted for completeness. Of course $\tau$, $\tau'$ approach infinity consistently with (14.10), and we use the conventional sign and delta function.) The square root will be principal, hence consistent with $\tau \to \infty$, $\tau' \to \infty$ along the positive and negative imaginary axes according to (14.10).

If we compare the asymptotic behaviors as applied to (14.11), (14.12), (14.13), (14.14), we find

$$H_{d+2,c}(\alpha/\beta) = H_{d,c+2}(\alpha/\beta) = H_{d,c}(\alpha/\beta), \tag{15.11}$$

$$H_{d\,c}(\epsilon^2\alpha/\beta) = H_{d,c}(\alpha/\beta), \tag{15.12}$$

$$H_{d,c}(\alpha/\beta + \eta) = e(c^2\eta^3/4)H_{d+1,c}(\alpha/\beta), \tag{15.13}$$

$$H_{d,c}(\alpha/\beta + 1) = e(c^2\eta^2/4)H_{d+c,c}(\alpha/\beta), \tag{15.14}$$

which incidentally could be verified from (15.7). A more difficult result is the *reciprocity*:

(15.15) $\quad H_{c,d}(\alpha/\beta)/N(\beta)^{\frac{1}{2}} = \operatorname{sgn}(\alpha,\beta)\,(H_{d,c}(-\beta/\alpha)/N(\alpha)^{\frac{1}{2}})\,e(cd\eta^2/2).$

Here

$$(15.16)\qquad \begin{cases} \operatorname{sgn}(\alpha,\beta) = -1 \text{ with the following arrays:} \\ \quad \begin{array}{cccc} \alpha & \alpha' & \beta & \beta' \\ \pm & \pm & \mp & \pm \\ - & + & \pm & \pm \end{array} \\ \operatorname{sgn}(\alpha,\beta) = +1, \text{ otherwise.} \end{cases}$$

The reciprocity relation (15.16) is proved by the Cauchy-Hecke method of comparing (15.10) with

$$(15.17)\qquad \Theta(c,d;\sigma) \approx H_{c,d}(-\beta/\alpha)\operatorname{sgn} N(\alpha)/N(\alpha)^{\frac{1}{2}}N(\alpha\sigma+\beta)^{\frac{1}{2}}$$

using (14.15) under the comparison $\sigma\tau = -1$. The details are very similar to Hecke [18] except that we permit $N(\alpha)^{\frac{1}{2}}$ to be imaginary, thereby forcing the "balancing" factor, $\operatorname{sgn}(\alpha,\beta)$, to be real.

By virtue of identities (15.11) through (15.14) we can confine all our attention to $H_{0,0}(\alpha/\beta)$ henceforth abbreviated:

$$(15.18)\qquad H_{0,0}(\alpha/\beta) = H(\alpha/\beta) = \sum_{\nu \bmod \beta} e(\nu^2\alpha/\beta).$$

Unless

$$(15.19)\qquad \alpha\beta \in \mathfrak{O}_2,$$

we note $H(\alpha/\beta)$ is trivially taken as 0.

**16. Hecke's modular function.** The superposition of the elements (15.10) for $\Theta^k(d,c;\tau)$ does not create an absolutely convergent series until $k > 4$. To remedy this fact, Hecke created the series with convergence factor:

$$(16.1)\qquad \begin{aligned} &\Psi(d,c;\tau;k,s) \\ &= \delta_{0,c} + \sum_{\alpha/\beta} H^k{}_{d,c}(\alpha/\beta)\operatorname{sgn}^k N(\beta)/N(\beta)^{k/2}N(\beta\tau-\alpha)^{k/2}\,|N(\beta\tau-\alpha)|^s. \end{aligned}$$

Our interest shall center about $k = 3$, but the general integer $k$ is momentarily useful. This series converges when

$$(16.2)\qquad k/2 + \operatorname{Re} s > 2.$$

(We shall be spared the tedium of repeating the convergence majorants by virtue of the similarity with estimates valid for $D = 5$; see [21].)

By virtue of the relations (15.11) through (15.15) we find, omitting symbols $k$, $s$ when convenient,

$$(16.3)\qquad \Psi(d,c;\tau) = \Psi(d+2,c;\tau) = \Psi(d,c+2;\tau),$$

(16.4) $$\Psi(d,c;\epsilon^2\tau)=\Psi(d,c;\tau),$$

(16.5) $$\Psi(d,c;\tau+\eta)=\Psi(d,c;\tau)e^k(c^2\eta^3/4),$$

(16.6) $$\Psi(d,c;\tau+1)=\Psi(d+c,c;\tau)e^k(c^2\eta^2/4),$$

(16.7) $$\Psi(d,c;-1/\tau)=\Psi(c,d;\tau)e^k(cd\eta^2/2)N(\tau)^{k/2}\,|N(\tau)|^s.$$

Hecke's work [7] shows that the function $\Psi$ has no finite singularities in $s$, hence by analytic continuation we can let $s=0$. In so doing, we obtain a function $\Psi$ satisfying the same functional equations as $\Theta^k$. The significance of this fact we leave until later on. Most important for computational purposes is the fact that the series (16.1) is susceptible to a fortuitous arrangement before making $s=0$. Again we spare the reader the details on majorants, by referring to the similar case where $D=5$.

It suffices to restrict ourselves to the case $d=0$, $c=0$ and simply write

(16.8) $$\Psi(\tau)=\Psi(0,0;\tau;k,s).$$

**17. Rearrangement of series.** We first write

(17.1) $$\Psi(\tau)=1+\sum_{\substack{\alpha/\beta \bmod 2\\ \alpha\beta\in\mathfrak{D}_2}}(H^k(\alpha/\beta)/|N(\beta)|^{k+s})\Phi(\alpha/\beta,\tau),$$

where

(17.2) $$\Phi(\alpha/\beta,\tau)=\sum_{\nu}N(\tau-\alpha/\beta+2\nu)^{-k/2}\,|N(\tau-\alpha/\beta+2\nu)|^{-s}.$$

(Note that the $\operatorname{sgn}^k N(\beta)$ of (16.1) disappears when $N(\beta)$ is divided out.) By making use of the Poisson-Lipschitz formula, again, we find

(17.3) $$\Phi(\alpha/\beta,\tau)$$
$$=(1/4D^{\frac12})\sum_{\mu}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e(-P\mu)\,dP\,dP'/N(P-\alpha/\beta+\tau)^{k/2}\,|N(P-\alpha/\beta+\tau)|^s,$$

where $P$ and $P'$ are real variables and $\mu$ is summed over $\mathfrak{D}$. Introducing, with Hecke,

(17.4) $$B(\mu,\tau)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e(-P\mu)\,dP\,dP'/N(P+\tau)^{k/2}\,|N(P+\tau)|^s,$$

we find the singular series

(17.5) $$\Psi(\tau)=1+\sum_{\mu}(B(\mu,\tau)/2D^{\frac12})P(\mu),$$

where

(17.6) $$P(\mu)=\sum_{\substack{\alpha/\beta \bmod 2\\ \alpha\beta\in\mathfrak{D}_2}}H^k(\alpha/\beta)e(-\mu\alpha/\beta)/2\ \ |N(\beta)|^{k+s}.$$

Now we know, in advance, that only totally positive $\mu$ can be present in $\Theta^k$, (although other $\mu$ are present before we set $s=0$). We can see more clearly that all $\mu$ entering into (17.5) are in $\mathfrak{O}_2$, for, by (15.14),

$$H(\alpha/\beta+1)=H(\alpha/\beta); \tag{17.7}$$

but $e(-\mu[\alpha/\beta+1])=-e(-\mu\alpha/\beta)$ if $\mu\notin\mathfrak{O}_2$.

We next introduce the symbol

$$\begin{aligned} P(\beta,\mu) &= \sum_{\substack{\alpha \bmod \beta, (\alpha,\beta)=1 \\ \alpha\beta\in\mathfrak{O}_2}} H^k(\alpha/\beta)e(-\mu\alpha/\beta)+H^k((\alpha+\beta\eta)/\beta)e(-\mu[\alpha+\beta\eta]/\beta), \end{aligned} \tag{17.8}$$

aware that $H(\alpha^*/\beta^*)=0$ if $\alpha^*\beta^*\notin\mathfrak{O}_2$. Then we find

$$P(\mu)=\sum_{(\beta)}P(\beta,\mu)/|N(\beta)|^{k+s}. \tag{17.9}$$

The remarkable simplification now achieved is due entirely to the fact that if $(\beta_1,\beta_2)=1$,

$$H(\alpha/\beta_1\beta_2)=H(\alpha_1/\beta_1)H(\alpha_2/\beta_2) \text{ for special } \alpha_1,\alpha_2, \tag{17.10}$$

$$P(\beta_1,\mu)\cdot P(\beta_2,\mu)=P(\beta_1\beta_2,\mu). \tag{17.11}$$

A corresponding result was proved by Maass [22; p. 736] using a Lemma of Siegel. Actually, a direct proof (like the one in the case of elementary Gaussian sums [20]) can work if we strictly observe that we must have $\alpha\beta\in\mathfrak{O}_2$, in our choice of $\alpha_1$ and $\alpha_2$ in (17.10).

Thus if we list the prime divisors in $R(D^{\frac{1}{2}})$, denoted by $\mathfrak{p}$, we find, by unique factorization,

$$P(\mu)=\prod_{\mathfrak{p}}P_{\mathfrak{p}}(\mu), \tag{17.12}$$

where,

$$P_{\mathfrak{p}}(\mu)=1+\sum_{t=1}^{\infty}P(\mathfrak{p}^t,\mu)/|N(\mathfrak{p})|^{t(k+s)}. \tag{17.13}$$

On the other hand, as $s\to 0$, when $\mu\gg 0$,

$$B(\mu,\tau)\to[4N(\mu)^{k/2-1}\pi^k/D^{k/2-1}\Gamma^2(k/2)]\,e(\mu\tau) \tag{17.14}$$

by use of the $\Gamma$-integral. Thus, making use of uniform convergence majorants [21] as $s\to 0$, we find, when $k=3$

$$\begin{cases}\Psi(\tau)=\sum\limits_{0\ll\mu\in\mathfrak{O}_2}e(\mu\tau)Z(\mu),\\ Z(\mu)=8\pi^2P(\mu)\,(N(\mu)/D^2)^{\frac{1}{2}}.\end{cases} \tag{17.15}$$

We have to next show $\Theta^3 = \Psi$ and then to simplify $P(\mu)$ to the point where it is recognizable. We do the latter first.

## Quadratic Residue Properties.

**18. Basic formulas.** It is apparent, by now, that we can deduce most of the results on $H(\alpha/\beta)$ by analogy with Gaussian sums. We can, of course, restrict ourselves to $\beta$ a prime power.

First we take $\pi$ an *odd* prime in $R(D^{\frac{1}{2}})$, $(\pi) \neq (\eta)$. We note $\pi = q$ if $(D/q) = -1$, for $q$ a rational prime and $\pi\pi' = p$ if $(D/p) = 1$, for $p$ a rational prime. We now standardize $\pi$ by the condition $\pi \in \mathfrak{O}_2$, at least to within the factor $\pm \epsilon^{2u}$, ($u$ integral), (by choosing $\epsilon\pi$ if $\pi \notin \mathfrak{O}_2$). Thus, for $D = 8$, $\pi = 3, 5, 1 \pm 2 \cdot 2^{\frac{1}{2}}, 11, 13, 5 \pm 2 \cdot 2^{\frac{1}{2}}, \cdots$, etc. For $D = 12$, $\pi = 3 + 2 \cdot 3^{\frac{1}{2}}, 5, 7, 1 \pm 2 \cdot 3^{\frac{1}{2}}, 5 \pm 2 \cdot 3^{\frac{1}{2}}$, etc. (Note, $(3 - 2 \cdot 3^{\frac{1}{2}}) = -(3 + 2 \cdot 3^{\frac{1}{2}})\epsilon^{-2}$.) The primes, of course, will not necessarily have positive norm.

Next we observe a "polygon" theorem:

$$(18.1) \qquad \sum_{\zeta \bmod \beta} e(2\zeta\alpha/\beta) = \begin{cases} |N(\beta)| & \text{if } \alpha/\beta \text{ is an integer,} \\ 0 & \text{otherwise.} \end{cases}$$

The proof consists of showing if $\alpha/\beta$ is no integer, the sum is unchanged by multiplication by $e(2\zeta_0\alpha/\beta)$, ($\neq 1$ for a properly chosen $\zeta_0$). Using this, we consider

$$(18.2) \qquad H(\alpha/\pi^t) = \sum_{\nu \bmod \pi^t} e(\nu^2\alpha/\pi^t), \qquad \alpha \in \mathfrak{O}_2,$$

We set $\nu = \pi^{t-1}\nu_1 + \nu_2$, ($\nu_1 \bmod \pi, \nu \bmod \pi^{t-1}$), and we discover

$$H(\alpha/\pi^t) = \sum_{\nu_2 \bmod \pi^{t-1}} e(\nu_2{}^2\alpha/\pi^t) \sum_{\nu_1 \bmod \pi} e(2\nu_1\nu_2\alpha/\pi),$$

$$(18.3) \qquad H(\alpha/\pi^t) = |N(\pi)| \, H(\alpha/\pi^{t-2}), \qquad \alpha \in \mathfrak{O}_2, t \geqq 2.$$

Since $H(\alpha) = 1$, trivially, we concentrate on $H(\alpha/\pi)$.

We next distinguish the quadratic residues mod $\pi$ by introducing the symbol

$$(18.4) \qquad (\lambda/\pi) = \begin{cases} 1 & \text{if } \lambda \equiv \xi^2 \bmod \pi, (\lambda, \pi) = 1, \\ -1 & \text{if } \lambda \not\equiv \xi^2 \bmod \pi, (\lambda, \pi) = 1, \\ 0 & \text{if } (\lambda, \pi) \neq 1. \end{cases}$$

Then if we consider $\rho$ the residues mod $\pi$, $(\rho/\pi) = 1$, and $\nu$ the non-residues mod $\pi$, $(\nu/\pi) = -1$, we form

$$(18.5) \qquad \begin{cases} R = \sum e(\rho/\pi), \\ N = \sum e(\nu/\pi), \end{cases}$$

since we can easily make $\rho \in \mathfrak{O}_2$ (by adding $\pi\epsilon$). Clearly

$$H(1/\pi) = 1 + 2R, \tag{18.6}$$

$$0 = 1 + R + N, \tag{18.7}$$

the latter following if we select $\rho$ and $\nu$ from the complete set of residues $2\zeta$, ($\zeta$ varying mod $\pi$), using (18.1). Thus, as in the Gaussian case,

$$H(1/\pi) = R - N = \sum (\alpha/\pi) e(\alpha/\pi), \tag{18.8}$$

$$H(\alpha/\pi) = (\alpha/\pi) H(1/\pi), \text{ if } \alpha \in \mathfrak{O}_2 \tag{18.9}$$

We now need to evaluate only $H(1/\pi)$. By quadratic reciprocity (15.15);

$$H(1/\pi)/N(\pi)^{\frac{1}{2}} = \operatorname{sgn}(1,\pi) H(-\pi/1)/1 = \operatorname{sgn}(1,\pi). \tag{18.10}$$

Hence, for odd $t$,

$$H(\alpha/\pi^t) = \{(\alpha/\pi)\operatorname{sgn}(1,\pi)\}^t \mid N(\pi)\mid^{t/2}\{\operatorname{sgn} N(\pi)\}^{\frac{1}{2}}, \tag{18.11a}$$

while, for even $t$,

$$H(\alpha/\pi^t) = \mid N(\pi)\mid^{t/2}. \tag{18.11b}$$

Next take the even prime, $\eta$. For $\beta = \eta^t$, we find that the expression $H(\alpha/\beta)$ makes sense, with $(\alpha,\beta) = 1$, exactly when $t \geqq 2$. Even so, using the polygon theorem (18.1), we find by an argument similar to the one in the last section,

(18.12) $H(\alpha/\eta^t) = 2H(\alpha/\eta^{t-2})$ if $t \geqq 4$. Here $\alpha$ need not belong to $\mathfrak{O}_2$. Thus every value of $H(\alpha/\eta^t)$ can be computed directly from these:

$$\begin{cases} H(1/\eta^2) = 2;\ H(1/\eta^3) = 2 \cdot 2^{\frac{1}{2}} i, \\[2mm] H((1+\eta)/\eta^2) = \begin{pmatrix} 2i \\ 2 \end{pmatrix},\ H(3/\eta^3) = 2 \cdot 2^{\frac{1}{2}} i, \\[2mm] H((1+\eta)/\eta^3) = \begin{pmatrix} 2 \cdot 2^{\frac{1}{2}} \\ 2 \cdot 2^{\frac{1}{2}} i \end{pmatrix},\ H((3+\eta/\eta^3) = \begin{pmatrix} -2 \cdot 2^{\frac{1}{2}} \\ -2 \cdot 2^{\frac{1}{2}} i \end{pmatrix}. \end{cases} \tag{18.13}$$

In cases where two values $\begin{pmatrix} \cdots \\ \cdots \end{pmatrix}$ are bracketed, the upper one belongs to $D = 8$ and the lower one belongs to $D = 12$.

We also note the further results:

$$H(\alpha/\eta^t + \eta) = H(\alpha/\eta^t), \qquad t \geqq 4, \tag{18.14}$$

$$H(\alpha/\eta^2 + \eta) = H(\alpha/\eta^2) e(\alpha), \tag{18.15}$$

$$H(\alpha/\eta^3 + \eta) = -H(\alpha/\eta^3). \tag{18.16}$$

The first follows from the observation that in the sum (15.18) for $H(\alpha/\eta^t)$ only the even $\nu$ enter, the odd $\nu$ cancel out to make possible (18.12).

By formula (17.10) we can show, finally,

$$|H(\alpha/\beta)| = |N(\beta)|^{\frac{1}{2}} \tag{18.17}$$

if $\alpha\beta \in \mathfrak{O}_2$ and $\alpha/\beta$ is reduced.

**19. Quadratic reciprocity.** The epitome of the theory is the deduction of quadratic reciprocity by the method of Hecke. Our principal variation consists in the restriction $\alpha\beta \in \mathfrak{O}_2$, in $H(\alpha/\beta)$.

If $\pi_1$ and $\pi_2$ are odd primes in $\mathfrak{O}_2$ by using (15.15), (18.9), and (18.10):

$$(\pi_1/\pi_2)(\pi_2/\pi_1) = \{\pi_1, \pi_2\}, \tag{19.1}$$

where $\{\pi_1/\pi_2\} = \operatorname{sgn}(1, \pi_1)\operatorname{sgn}(1, \pi_2)\operatorname{sgn}(\pi_1, \pi_2)\operatorname{sgn} N(\pi_1)$, or

$$\tag{19.2} \begin{cases} \{\pi_1, \pi_2\} = -1 \text{ under the array of signs:} \\ \pi_1 \quad \pi_1' \quad \pi_2 \quad \pi_2' \\ \pm \quad \mp \quad \pm \quad \mp, \quad \pi_1\pi_2 \gg 0; N(\pi_1) < 0, N(\pi_2) < 0; \\ \pm \quad \mp \quad - \quad -, \quad \pi_2 \ll 0; N(\pi_1) < 0; \\ - \quad - \quad \pm \quad \mp, \quad \pi_1 \ll 0; N(\pi_2) < 0; \\ \{\pi_1, \pi_2\} = +1, \text{ otherwise.} \end{cases}$$

Then the first completion theorem also emerges in the process:

$$(-1/\pi) = \operatorname{sgn} N(\pi). \tag{19.3}$$

We can deduce the rational results: If $\pm p$ denotes $\pi\pi'$, $p > 0$, then $\xi^2 + 1 \equiv 0 \bmod \pi$ is solvable exactly when $x^2 + 1 \equiv 0 \bmod p$ is solvable. This implies, in rational symbols, $(D/p) = (-1/p) = +1$, hence $p \equiv 1 \bmod D$ for $D = 8$ and $12$. Thus the elementary theory of quadratic forms yields the following expression, for an arbitrary $p \equiv 1 \pmod 8$:

$$p = u_1^2 - 8v_1^2 = N(\pi_1), \qquad \pi_1 = u_1 + 2 \cdot 2^{\frac{1}{2}} v_1, \tag{19.4}$$

and one (not both) of the following for $p \equiv 1 \pmod{12}$:

$$p = \begin{cases} u_2^2 - 12v_2^2 = N(\pi_2), & \pi_2 = u_2 + 2 \cdot 3^{\frac{1}{2}} v_2, \\ 4u_3^2 - 3v_3^2 = N(\pi_3), & \pi_3 = 2u_3 + 3^{\frac{1}{2}} v_3. \end{cases} \tag{19.5}$$

Clearly, $\pi_1$, $\pi_2$, and $(2 + 3^{\frac{1}{2}})\pi_3$ all $\in \mathfrak{O}_2$, verifying (19.3). Conditions (19.4) and (19.5) are clearly necessary and sufficient for $(-1/\pi) = 1$, when $\pi$ is not a rational prime.

A less simple case is the second completion theorem. We apply the reciprocity formula (15.15) to $H(\eta^3/-\pi)$ and obtain a set of values of

$(\eta/-\pi)$ as related to $H(+\pi/\eta^3)$, which is determined by $\pi \pmod{2\eta^3}$, (not mod $\eta^3$, by (18.16)). Since the steps are finite in number, we omit tedious details:

$$(19.6) \qquad (\eta/\pi) = \operatorname{sgn}(1, -\pi)\operatorname{sgn}(\eta^3, -\pi) H(\pi/\eta^3)/2\ 2^{\frac{1}{2}}, \qquad (\pi \in \mathfrak{O}_2).$$

Thus from formulas (18.13) and (18.16) for $D = 8$ or $12$:

$$(19.7) \qquad (\eta/\pi) = \begin{cases} +\operatorname{sgn}\pi' \text{ if } \pi \equiv 1, 1+\eta^3, 3, 3+\eta^3 \pmod{4\eta}, \\ -\operatorname{sgn}\pi' \text{ if } \pi \equiv 5, 5+\eta^3, 7, 7+\eta^3 \pmod{4}. \end{cases}$$

The final completion law is found by doing likewise with $H(\epsilon\eta^3/-\pi)$ and finding $(\epsilon\eta/\pi)$. Omitting details, we note

$$(19.8) \qquad (\epsilon\eta/\pi) = \operatorname{sgn}(1, -\pi)\operatorname{sgn}(\epsilon\eta^3, -\pi) H(\pi/\epsilon\eta^3)/2\ 2^{\frac{1}{2}}[-N(\epsilon)]^{\frac{1}{2}}, \qquad (\pi \in \mathfrak{O}_2).$$

Thus, as before, with $\epsilon$ defined in (14.2), we find the remarkable result for $D = 8, \pi \in \mathfrak{O}_2$,

$$(19.9) \qquad (\epsilon/\pi) = \operatorname{sgn}\pi' \text{ if } \pi \equiv 1, 3+2\eta \qquad \pmod{4},$$

$$(19.10) \qquad (\epsilon/\pi) = -\operatorname{sgn}\pi' \text{ if } \pi \equiv 1+2\eta, 3 \qquad \pmod{4},$$

and for $D = 12, \pi \in \mathfrak{O}_2$,

$$(19.11) \qquad (\epsilon/\pi) = 1 \text{ if } \pi \equiv 1, 3 \qquad \pmod{4},$$

$$(19.12) \qquad (\epsilon/\pi) = -1 \text{ if } \pi \equiv 1+2\eta, 3+2\eta \qquad \pmod{4}.$$

We note that lines (19.10) and (19.11) represent the odd residue classes for which $\pi \equiv -\xi^2 \bmod 4$. The pattern emerges again for the table in § 12 (above).

**20. The zeta series.** We next consider the various series $P\mathfrak{p}(\mu)$, starting with $P(\mathfrak{p}^t, \mu)$, of § 17.

If $\pi$ is odd, in expression (17.8) for $P(\pi^t, \mu)$ we note if $\alpha$ and $\pi \in \mathfrak{O}_2$, then we have simply (for $k = 3$),

$$(20.1) \qquad \begin{aligned} P(\pi^t, \mu) & \sum_{\substack{\alpha \bmod \pi^t \\ (\alpha,\pi)=1}} H^3(\alpha/\pi^t) e(-\mu\alpha/\pi^t), \qquad t \geqq 1) \\ &= |N(\pi)|^{3t/2}[\operatorname{sgn}^t N(\pi)]^{3/2} \operatorname{sgn}^t(1, \pi) \sum_{\alpha \bmod \pi^t} (\alpha/\pi)^t e(-\mu\alpha/\pi^t). \end{aligned}$$

The inner sum is recognizable again as a character sum, not unlike (18.8), except for the fact that $\pi$ could divide $\mu$. Let us write

$$(20.2) \qquad \mu = \pi^m \mu^* \qquad (\mu^*, \pi) = 1, \qquad (m \geqq 0),$$

then we find after an elaborate effort:

$$(2.3)\quad P(\pi^t,\mu)=\begin{cases} \left.\begin{matrix} 0 & t\ \text{odd} \\ |N(\pi)|^{5/2}(1-1/|N(\pi)|) & t\ \text{even}\end{matrix}\right\} & 1\leqq t\leqq m, \\ \left.\begin{matrix} |N(\pi)|^{(5t-1)/2}(-\mu^*/\pi) & t\ \text{odd} \\ -|N(\pi)|^{5t/2-1} & t\ \text{even}\end{matrix}\right\} & t=m+1, \\ 0 & t>m+1.\end{cases}$$

The interesting cases are where $\mu$ is square-free so $m=0$ or 1. In either case, following a familiar pattern [12, p. 190]

$$(20.4)\qquad P_\pi(\mu)=(1-|N(\pi)|^{-2-2s})/(1-(-\mu/\pi)|N(\pi)|^{-1-s}),\quad \pi\neq\eta.$$

The contribution of $\eta$ is more complicated. First of all, the analogue of (20.1) is the more complex expression

$$(20.5)\qquad \begin{aligned} P(\eta^t,\mu)=\sum_{\substack{\alpha(\operatorname{mod}\eta^t)\\(\alpha,\eta)=1}} & H^3(\alpha/\eta^t)e(-\mu\alpha/\eta^t) \\ & +H^3(\alpha/\eta^t+\eta)e(-\mu\alpha/\eta^t-\mu\eta).\end{aligned}$$

We therefore see that if $\mu$ is odd, $\mu\in\mathfrak{O}_2$, by (18.14), $P(\eta^t,\mu)\equiv 0$ if $t\geqq 4$. The problem is then finitary and details can be omitted. We introduce the characters

$$(20.6)\qquad Q(-\mu,\epsilon)=\begin{cases} 1 \text{ if } -\mu\equiv\xi^2 \bmod 4, \\ -1 \text{ if } -\mu\not\equiv\xi^2 \bmod 4.\end{cases}$$

(Clearly $Q=-1$ necessarily if $\mu\notin\mathfrak{O}_2$). Since $\mu\gg 0$, this character is consistent with (19.9), etc., (if $\pi=-\mu$ formally). We introduce the further character,

$$(20.7)\qquad Q(-\mu,\eta)=\begin{cases} 1 \text{ if } -\mu\equiv\xi^2 \pmod{4\eta},\ \mu \text{ odd}; \\ -1 \text{ if } -\mu\not\equiv\xi^2 \pmod{4\eta} \text{ but } -\mu\equiv\xi^2 \pmod 4,\ \mu \text{ odd}; \\ 0 \text{ if } -\mu\not\equiv\xi^2 \bmod 4, \text{ or } (\mu,\eta)\neq 1.\end{cases}$$

The characters $Q(-\mu,\epsilon)$ and $Q(-\mu,\eta)$ are like the rational characters $(-1/p)$ and $(p/2)$.

Now we can verify that if $\mu$ is square-free, and $\mu\gg 0$, $\mu\in\mathfrak{O}_2$,

$$(20.8)\qquad P(\eta^2,\mu)=16Q(-\mu,\epsilon),$$

$$(20.9)\qquad P(\eta^3,\mu)=128Q(-\mu,\eta),$$

so that for $\mu\in\mathfrak{O}_2$, square-free, totally positive, $P_\eta(\mu)=P_0(\mu)$, where

$$(20.10)\qquad P_0(\mu)=1+Q(-\mu,\epsilon)/4\ \ 2^{2s}+Q(-\mu,\eta)/4\ \ 2^{3s}.$$

We more generally consider $\mu\gamma^2$ defined according to (12.3) to cover cases where $\mu\notin\mathfrak{O}_2$. Then it can be seen

$$(20.11)\qquad P_\eta(\mu\gamma^2)=P_0(\mu).$$

We consider terms of the special type with coefficient defined as shown:

$$(20.12)\qquad \Psi(\tau)=\sum e(\mu\gamma^2\tau)Z(\mu\gamma^2)+\cdots.$$

We then make a series of substitutions to obtain a convenient formula for $Z(\mu\gamma^2)$. The details are lengthy, but we might just dwell on properties of the field $R(D^{\frac{1}{2}}, (-\mu)^{\frac{1}{2}})$ as a relative quadratic field over $R(D^{\frac{1}{2}})$. Thus the zeta function

$$(20.13)\qquad \prod_{\pi}(1-|N(\pi)|^{-2})^{-1}=16\pi^4C/D^{3/2},\quad (C_8=1/48, C_{12}=1/24),$$

from § 5. We also note the relative zeta function [18] for the adjunction of $(-\mu)^{\frac{1}{2}}$ to $R(D)^{\frac{1}{2}}$ is

$$(20.14)\qquad \begin{aligned}&\prod_{\pi\text{ odd}}(1-(-\mu/\pi)/|N(\pi)|)^{-1}\cdot(1-Q(-\mu,\eta)/2)^{-1}\\ &\qquad =H(D^{\frac{1}{2}},-\mu)^{\frac{1}{2}})(2\pi^2/w)/(\log|\epsilon|/\log|E|)(\Delta/D)^{\frac{1}{2}},\end{aligned}$$

where $\Delta$ is the discriminant of $R(D^{\frac{1}{2}}, (-\mu)^{\frac{1}{2}})$, $w$ is the number of complex roots of unity and $E$ is the fundamental unit of this larger field. We can now write, if we let $s\to 0$, in the terminology of (12.2),

$$(20.15)\qquad Z(\gamma^2\mu)=H(D^{\frac{1}{2}}, (-\mu)^{\frac{1}{2}})G$$

where, for square-free, totally positive $\mu$,

$$(20.16)\qquad G=\frac{\left\{1+\dfrac{Q(-\mu,\epsilon)}{4}+\dfrac{Q(-\mu,\eta)}{4}\right\}\ (4/3)\left\{1-\dfrac{Q(-\mu,\eta)}{2}\right\}\ (2/w)}{C_D(\Delta/N(\mu\gamma^2)D^2)^{\frac{1}{2}}\ \dfrac{\log|\epsilon|}{\log|E|}}$$

This is substantially the $G$ of (12.2), if we find $\Delta$ according to the usual theory of relative-quadratic fields. Referring to (20.6), the relative basis is (for $D=8, 12$),

$$(20.17)\qquad \begin{cases}[1, (\xi+(-\mu)^{\frac{1}{2}})/2], & \text{if } Q(-\mu,\epsilon)=1,\\ [1, (1+(-\mu)^{\frac{1}{2}})/\eta], & \text{if } Q(-\mu,\epsilon)=-1,\ \mu\in\mathfrak{O}_2,\\ [1, (-\mu)^{\frac{1}{2}}], & \text{if } \mu\notin\mathfrak{O}_2.\end{cases}$$

Of course, in the first two cases $\gamma=1$, while in the third $N(\gamma)=2$. To complete the explanation, we might add that $w=2$ and $|\epsilon|=|E|$ except in cases cited in (12.4), which we exclude.

As a concluding remark to this section we observe that the progress from formulae (17.12-17.15) to (20.15) depends heavily on Gaussian sum manipulations. Here many details were not needed, particularly those typified by the

reduction of such sums to "normal form" and the use of the "zero-rule," as, for example, in Hasse's work [16a; p. 14]. The zeta-function manipulations, however, are still best derived as *ad hoc* extensions of Maass' work [12]; while the identification of the modular forms will be done by a more direct method, starting in § 22 below.

**21. Singular series for four squares.** Before proceedings to the proof that $\Theta^3 = \Psi$, we note in retrospect how the singular series could be developed for the fourth power. In the earlier paper, [16], formula (51a) yields, essentially, for $D = 8, 12$, that the singular series for $\Theta_{0,0}(\tau)$ is

$$\Omega_{0,0}(\tau) = \{4A_1(\tau) + 64A_0(4\tau) + 4A_0(\tau) - 24A_0(2\tau)\}C_D' + 1 \tag{21.1}$$

where

$$\begin{cases} A_\kappa(\tau) = \lim\limits_{s\to\infty} \sum [N(\nu\tau+\mu)]^{-2} \,|\, N(\nu\tau+\mu)\,|^{-s}, \\ \nu\kappa \equiv \mu, \ (\mathrm{mod}\ 2), \\ (\nu) \neq (0), \end{cases} \tag{21.2}$$

and

$$C_D' = D^{3/2}/48\pi^4 C_D. \tag{21.3}$$

It is no great difficulty to evaluate the singular series (16.1) for $k = 4$ and obtain the matching series

$$\Psi(0,0;\tau;4,s) = 1 + \sum_{\alpha/\beta,\alpha\beta\in\mathfrak{D}_2} [N(\beta\tau-\alpha)]^{-2} \,|\, N(\beta\tau-\alpha)\,|^{-s}. \tag{21.4}$$

This can best be done using formulas (17.10), (18.11), (18.13), which assures us

$$H^4(\alpha/\beta) = N(\beta)^2. \tag{21.5}$$

We pause here to note that the summation condition "$\alpha\beta \in \mathfrak{D}_2$ for $\alpha/\beta$ reduced" actually encompasses the rather complicated coefficients in (21.1). In fact, examining Götzky's work equally critically we note, for $D = 5$,

$$[\sum e(\nu^2\tau)]^4 = \{8A_0(\tau) + 128A_0(4\tau) - 16A_0(2\tau)]D^{3/2}/\pi^4 + 1. \tag{21.6}$$

Again, this fairly complicated expression of Götzky could have been written as (21.4), $(s \to 0)$, with the restriction $\alpha/\beta$ be reduced and $\alpha$ and $\beta$ not be both odd. (The last restriction, of course, enters Jacobi's well-known demonstration involving four *rational* squares.)

## Equality of Theta Series and Singular Series.

**22. Modular forms.** The functions $\Psi(d,c;\tau,3,0)$ and $\Theta^3(d,c;\tau)$ satisfy the same functional equations (14.11)-(14.15) and (16.3)-(16.7). Are they proportional?

We revert to the procedure of § 7. We note that $\Theta(d,c;\tau)$ can vanish, within its functional domain, only on a well defined zero-manifold. This manifold exists only for $d=c=1$ and is the complex (two dimensional) manifold

$$v=0 \tag{22.1}$$

in new complex cooridnates $(u,v)$ defined as follows:

$$\begin{cases} \tau=-\epsilon' u+v, \\ \tau'=-\epsilon u+v, \text{ for } D=8; \end{cases} \tag{22.2}$$

$$\begin{cases} \tau=u+v, \\ \tau'=-u+v, \text{ for } D=12. \end{cases} \tag{22.3}$$

Furthermore, on this manifold, the zeros are simple, (see § 7), or,

$$\partial\Theta(u,v)/\partial v\neq 0 \text{ for all } u, \text{ (at } v=0). \tag{22.4}$$

We need only show that $\Psi(d,c;\tau,3,0)/\Theta^3(d,c,\tau)$ has no poles in the fundamental domain by establishing a third order zero for $\Psi$ when $d=c=1$ and $v=0$. We can draw our conclusion (that $\Psi$ and $\Theta^3$ are proportional) from Theorem 4 (§ 2).

We define $\Xi(d,c;\tau)$ as a system of four modular functions (as $d$, $c$ vary) in $\tau$ and $\tau'$, having no finite singularities and satisfying the system (16.3)-(16.7) with $k=3$, (dimension $-3/2$), and $s=0$. Then we conclude

$$\begin{cases} \Xi(1,1,-\epsilon'^2/\tau^2)=\Xi(1,1,\tau)N(\tau)^{3/2}, \\ \Xi(1,1,\tau-\epsilon')=-i\Xi(1,1,\tau), \text{ for } D=8; \end{cases} \tag{22.5}$$

$$\begin{cases} \Xi(1,1,-1/\tau)=\Xi(1,1,\tau)N(\tau)^{3/2}, \\ \Xi(1,1,\tau+\eta-1)=-\Xi(1,1,\tau), \text{ for } D=12. \end{cases} \tag{22.6}$$

We next define

$$\begin{cases} \Gamma(u,v)=\Xi(1,1;\tau)=\Xi(1,1;\tau,\tau'), \\ \Gamma_t(u)=(\partial/\partial v)^t\Gamma(u,v) \text{ at } v=0. \end{cases} \tag{22.7}$$

Then let $j$ denote the integer $\geqq 0$, for which

$$\Gamma_0(u)\equiv\Gamma_1(u)\equiv\cdots\equiv\Gamma_j(u)\equiv 0;\ \Gamma_{j+1}(u)\not\equiv 0. \tag{22.8}$$

Then since $N(\tau)=-u^2$, on the zero manifold while $\partial/\partial v=\partial/\partial\tau+\partial/\partial\tau'$, etc., for $t\leqq j$,

$$(22.9)\qquad \begin{cases} \Gamma_t(u+1) = -i\Gamma_t(u), \\ \Gamma_t(-1/u) = -i\Gamma_t(u)u^{2t+3}, \\ \Gamma_t(u) \to 0, \text{ as } u \to i\infty, \text{ for } D=8; \end{cases}$$

$$(22.10)\qquad \begin{cases} \Gamma_t(u+3^{\frac{1}{2}}) = -\Gamma_t(u), \\ \Gamma_t(-1/u) = -i\Gamma_t(u)u^{2t+3}, \\ \Gamma_t(u) \to 0, \text{ as } u \to i\infty, \text{ for } D=12. \end{cases}$$

If we could show $j \geqq 3$, we shall then have established that $\psi = \Theta^3$. We can do this only for $D=8$ (and with an additional bit of special information, distinguishing this case from $D=12$, as we now develop).

**23. Special Fourier expansions.** We first shall estimate the order of magnitude of $\Gamma_t(u)$ as $u \to \infty$. To do this we expand $\psi(1,1;\tau;k,s)$ by the Poisson-Lipschitz formula as in § 17. The only difference is that, whereas earlier $H(\alpha/\beta) = H_{0,0}(\alpha/\beta) = H_{0,0}(\alpha/\beta + 2\nu)$, now we must use

$$(23.1)\qquad H_{1,1}(\alpha/\beta + 2\nu) = e(\nu\eta^2/2)H_{1,1}(\alpha/\beta)$$

Thus we must insert the factor $e(\nu\eta^2/2)$ into the numerator of the fraction in the summation of (17.2). This produces the result (in contrast to (17.15)),

$$(23.2)\qquad \Psi(1,1;\tau,3,s) = \sum_{\mu^* >> 0} e(\mu^*\tau)8\pi^2 P^*(\mu)(N(\mu^*)D^{-2})^{\frac{1}{2}},$$

where

$$(23.3)\qquad \mu^* = \mu + \tfrac{1}{2}\cdot\eta^2/2, \qquad \mu \in \mathfrak{D}_2,$$

and

$$(23.4)\qquad P^*(\mu^*) = \tfrac{1}{2}\sum_{\alpha/\beta \bmod 2} H^3{}_{1,1}(\alpha/\beta)e(-\mu^*\alpha/\beta)\,|\,N(\beta)\,|^{-(k+s)}$$

where $H_{1,1}$ has meaning, of course, only when

$$(23.5)\qquad (\alpha+\eta)(\beta+\eta) \in \mathfrak{D}_2.$$

We now have to separate the cases: When $D=8$, we write, in the notation (22.2),

$$(23.6)\qquad \begin{cases} \Psi(1,1;\tau;3,s) \\ = \pi^2 \sum [N(\mu^*)]^{\frac{1}{2}} \exp \pi i u(a-b+\tfrac{1}{2}) \cdot \exp \pi i b v \cdot P^*(\mu^*), \end{cases}$$

with the restriction

$$(23.7)\qquad \mu^* = a + b2^{\frac{1}{2}} + \tfrac{1}{2} >> 0.$$

When $D=12$, we write in the notation (22.3)

$$(23.8)\qquad \begin{cases} \Psi(1,1;\tau;3,s) \\ = (2\pi^2/3)\sum [N(\mu^*)]^{\frac{1}{2}} \exp \pi i u(a+b)/3^{\frac{1}{2}} \cdot \exp \pi i(b+\tfrac{1}{2})v \cdot P^*(\mu^*) \end{cases}$$

with the restriction

$$\mu^* = (a+b) + (b+\tfrac{1}{2})3^{\frac{1}{2}} \gg 0. \tag{23.9}$$

Now if we are interested in the behavior of $\Gamma(u,v)$ in $u$, (as $u \to i\infty$), we must arrange $\mu^*$ in order of increasing values of $a - b + \frac{1}{2}$ when $D = 8$ and $a + b$ when $D = 12$. Thus, for $D = 8$, the first few terms are

$$\mu^* = 1/2, 3/2 + 2^{\frac{1}{2}}; 3/2, 5/2 + 2^{\frac{1}{2}}, 7/2 + 2\cdot 2^{\frac{1}{2}}, 9/2 + 3\cdot 2^{\frac{1}{2}}; \cdots;$$

and

$$\begin{aligned}(1/\pi^2)\Psi(1,1;\tau;3,s) &= \tfrac{1}{2}\exp(\pi iu/2)[P^*(\tfrac{1}{2}) + \exp \pi iv \cdot P^*(\tfrac{3}{2} + 2^{\frac{1}{2}})] \\ &+ \exp(3\pi iu/2)[3/2P^*(3/2) + 17^{\frac{1}{2}}/2P^*(5/2 + 2^{\frac{1}{2}})\exp \pi iv \\ &+ 17^{\frac{1}{2}}/2P^*(7/2 + 2\cdot 2^{\frac{1}{2}})\exp 2\pi iv + 3/2P^*(9/2 + 3\cdot 2^{\frac{1}{2}})\exp 3\pi iv] \\ &+ \cdots.\end{aligned} \tag{23.10}$$

For $D = 12$, the first few terms are

$$\mu^* = 1 + 3^{\frac{1}{2}}/2, 1 - 3^{\frac{1}{2}}/2; 2 + 3^{\frac{1}{2}}/2, 2 - 3^{\frac{1}{2}}/2; \cdots;$$

and

$$\begin{aligned}&(3/2\pi^2)\Psi(1,1;\tau;3,s) \\ &= \tfrac{1}{2}\exp(\pi iu/3^{\frac{1}{2}})[P^*(1 + 3^{\frac{1}{2}}/2) + \exp(-\pi iv/2)P^*(1 - 3^{\frac{1}{2}}/2)] \\ &+ 13^{\frac{1}{2}}/2\exp(2\pi iu/3^{\frac{1}{2}})[P^*(2 + 3^{\frac{1}{2}}/2) + \exp(-\pi iv/2)P^*(2 - 3^{\frac{1}{2}}/2)] \\ &+ \cdots.\end{aligned} \tag{23.11}$$

We next observe that for $D = 8$,

$$P^*(\mu) = 0 \text{ if } a \equiv b \bmod 2. \tag{23.12}$$

This is true because of a symmetry in the formula (23.4). For instance, if $\alpha/\beta$ has the property (23.5), so does $\alpha/\beta + (1+\eta)$. Now when $\alpha/\beta$ is augmented by $1+\eta$, $H^3{}_{1,1}(\alpha/\beta)e(-\mu^*\alpha/\beta)$ is multiplied by

$$i^3 e(-\mu^*[1+\eta]) = -e([a+b]\eta),$$

which equals $-1$ when $a \equiv b \bmod 2$. Thus all terms of type $\exp \pi i(4s-1)u/2$ vanish in (23.10), and, in particular, differentiating with respect to $v$ (at $v = 0$), we find, condition (22.9) supplemented by the strong result

$$\Gamma_t(u) = O(\exp 3\pi iu/2) \text{ as } u \to i\infty, \tag{23.13}$$

if we start by letting $\Xi = \Psi(d,c;\tau;3,0)$ in equation (22.7).

Turning our attention to $D = 12$, we are less fortunate;

$$P^*(\mu^*) = 0, \text{ if } a \not\equiv b \bmod 2, \tag{23.14}$$

since $H^3{}_{1,1}(\alpha/\beta)e(-\mu^*\alpha/\beta)$ becomes multiplied by $+e([a+b]\eta)$ when $\alpha/\beta$ is augmented by $1+\eta$. Thus all terms of type $\exp 2t\pi iu/3^{\frac{1}{2}}$ vanish in (23.11). Thus (22.10) can be supplemented only by the comparatively weak rsult

$$\Gamma_t(u) = O(\exp \pi iu/3^{\frac{1}{2}}), \tag{23.15}$$

if we start with $\Xi = \Psi$ in equation (22.7).

**24. Completion of the identity for $D = 8$.** We first take the case $D = 8$ where the rational modular group has the famous Klein invariant $J(u)$. The factors $-i$ in system (22.9) serves to forecast the presence of radicals. To simplfy matters, we take fourth powers:

$$\Gamma_t{}^4(u+1) = \Gamma_t{}^4(u), \tag{24.1a}$$

$$\Gamma_t{}^4(-1/u) = \Gamma_t{}^4(u)u^{8t+12}, \tag{24.1b}$$

$$\Gamma_t{}^4(u) = O[J^{-3}(u)] \text{ as } u \to i\infty, \tag{24.1c}$$

the last expression coming from (23.13). Thus

$$\Gamma_t{}^4(u) = J'(u)^{4t+6} f[J(u)]J(u)^{-t_1}[J(u)-1]^{-t_2}, \tag{24.2}$$

where, in the notation of "integral part" $[\cdot\ \cdot\ \cdot]$,

$$t_1 = 4 + [8t/3], \qquad t_2 = 2t + 3, \tag{24.3}$$

and the degree of $f$ is $\leqq t_1 + t_2 - 4t - 9$, by the usual method of Poincaré [2].

Beyond these considerations $\Gamma_t{}^4(u)$ must behave like a perfect fourth power near every finite complex value of $u$:

| $t$ | $t_1$ | $t_2$ | max degree $f$ |
|---|---|---|---|
| 0 | 4 | 3 | $-2$ |
| 1 | 6 | 5 | $-2$ |
| 2 | 9 | 7 | $-1$ |
| 3 | 12 | 9 | 0 |

Thus the first instance is, at $t = 3$,

$$\Gamma_3{}^4(u) = \text{const.}\, J'(u)^{18} J(u)^{-12}[J(u)-1]^{-9}. \tag{24.4}$$

The right-hand member indeed is a perfect fourth power. This will not *necessarily* imply that $\Gamma_3(u)$ transforms with the factors required in (22.9). (This requires continuation.) We have nevertheless established that in the

terminology of § 22, $j \geqq 3$ for $D = 8$. Thus $\Psi/\Theta^3$ is free of singularities and finally by Theorem 4,

$$\Psi(d, c; \tau; 3, 0) = \Theta^3(d, c; \tau), \tag{24.5}$$

$$A_3(\mu) = Z(\mu). \tag{24.6}$$

We can justify anew the assertion that

$$\Gamma_3^*(u) = J'(u)^{9/2}[J(u) - 1]^{-9/4}J(u)^{-3} \tag{24.7}$$

must necessarily satisfy the properties (22.9). For instance, such a $\Gamma_3(u)$ exists because $\Theta$ has only a simple zero and starting with $\Xi = \Psi$ in (22.7) we must necessarily arrive at a $\Gamma_3 \not\equiv 0$.

Continuing along the lines of the earlier part [16], we turn our attention to dimensionalities: The *modular* forms $\Xi(d, c; \tau)$ *described by equations* (16.3)-(16.7) *with* $k = 3$, $s = 0$, *are of dimension one* if they are bounded at infinity. To see this, we note that even in the absence of the special information in § 23, we can conclude that $\Gamma_t(u) = O(J^{-1/4}(u))$, from equations (22.9). Yet we could conceivably have more forms $\Gamma_t^4(u)$, since the degree of $f[J(u)]$ can now be $t_1 + t_2 - 4t - 7$. We still insist $\Gamma_t(u)$ must be single-valued in $u$ regardless of how it transforms under the modular group. We then find these additional forms for $\Gamma_0^4$, $\Gamma_2^4$, $\Gamma_3^4$ are the only possibilities:

$$J'(u)^6/J(u)^4[J(u) - 1]^3, \qquad J'(u)^{14}/J(u)^8[J(u) - 1]^7,$$
$$J'(u)^{18}/J(u)^{12}[J(u) - 1]^7.$$

Even though these are fourth powers, their fourth roots are of the form $[J(u) - 1]^{\frac{1}{4}}\Gamma_3^*(u)$ (Rat. Funct. of $J'$, $J$). Thus since $[J(u) - 1]^{\frac{1}{4}}$ acquires a sign change when $u \to u + 1$, it follows that the only solution to system (22.9) is still (24.7).

**25. Failure of the identity for $D = 12$.** Here we consider the transformation (22.10) for which Hecke's invariant $I(u)$ is most natural (see § 10). It has the properties

$$\left\{\begin{array}{l} I(u + 3^{\frac{1}{2}}) = I(u), \\ I(-1/u) = I(u), \\ I(u) = \text{const} \cdot \exp - 2\pi i u/3^{\frac{1}{2}} + O(1), \text{ as } u \to i\infty, \\ I(u) = \text{const} \cdot (u - i)^2 + 1 + O(u - i)^3, \text{ as } u \to i, \\ I(u) = \text{const} \cdot (u - \exp 2\pi i/6)^6 + O(u - \exp 2\pi i/6)^7, \\ \qquad\qquad \text{as } u \to \exp 2\pi i/6. \end{array}\right. \tag{25.1}$$

We note

$$(25.2)\qquad \begin{cases} \Gamma_t^4(u+3^{\frac{1}{2}})=\Gamma_t^4(u), \\ \Gamma_t^4(-1/u)=\Gamma_t^4(u)u^{8t+12}, \\ \Gamma_t^4(u)=O[I(u)^{-2}] \text{ as } u\to i\infty. \end{cases}$$

Then by the usual method

$$(25.3)\qquad \Gamma_t^4(u)=I'(u)^{4t+6}f[I(u)]I(u)^{-t_1}[I(u)-1]^{-t_2},$$

where

$$(25.4)\qquad t_1=5+[10t/3],\qquad t_2=2t+3$$

and $f$ is a polynomial of degree $\leqq t_1+t_2-4t-8$. We then obtain the values

| $t$ | $t_1$ | $t_2$ | max degree $f$ |
|---|---|---|---|
| 0 | 5 | 3 | 0 |
| 1 | 8 | 5 | 1 |
| 2 | 11 | 7 | 2 |
| 3 | 15 | 9 | 4 |

While there are now plenty of polynomials, more important, many are *fourth* powers in the $u$-plane. For $t=0,1,2$ we have the following exact fourth powers:

$$(24.5)\qquad \Gamma_0^*(u)^4=I'(u)^6I(u)^{-5}[I(u)-1]^{-3},$$

$$(24.6)\qquad \Gamma_1^*(u)^4=I'(u)^{10}I(u)^{-7}[I(u)-1]^{-5},$$

$$(24.7)\qquad \Gamma_2^*(u)^4=I'(u)^{14}I(u)^{-11}[I(u)-1]^{-5},$$

$$(24.8)\qquad \Gamma_2^{**}(u)^4=I'(u)^{14}I(u)^{-11}[I(u)-1]^{-7}.$$

We can verify that $\Gamma_0^*(u)\approx \text{const}+O(u-i)$ hence comparing behaviors at $u=i$, $\Gamma_0^*(u)$ does not satisfy the required sign condition, i.e., $\Gamma_0(-1/u)=+i\Gamma_0(u)u^3$. Moreover, $\Gamma_1^*$ and $\Gamma_2^*\approx \text{const}(u-i)$, hence they satisfy conditions (22.10), but $\Gamma_2^{**}$ must be excluded since $\Gamma_2^{**}(u+3^{\frac{1}{2}})=+\Gamma_2^{**}(u)$ from its order at $i\infty$. *Thus the modular forms* $\Xi(d,c;\tau)$ *described by equations* (16.3)-(16.7) *with* $k=3$, $s=0$ *are a vector space of dimension two, or three* if they are bounded at infinity.

We know that $\Psi(d,c;\tau;3,0)\neq\Theta^3(d,c;\tau)$ because $Z(1)=8\neq A_3(1)=6$, so the coefficients of $e(\tau)$ do not match when $d=c=0$. The third independent solution of the $\Xi$ system is not apparent, if it exists at all.

**26. Rational results.** If we take the coefficient in $\Psi$ for $e(\pi^2\mu\tau)$, namely $Z(\pi^2\mu)$, there is a very simple relation with $Z(\mu)$ in the case where

$\pi \in \mathfrak{O}_2$ is an odd prime not dividing the square-free totally positive $\mu$. Using formulae (20.3) and (20.4) we find

$$P_\pi(\mu\pi^2) = \begin{cases} P_\pi(\mu)(1+2)/|N(\pi)|), & N(\pi) > 0, \\ P_\pi(\mu), & N(\pi) < 0. \end{cases} \tag{26.1}$$

This is the only factor of $P(\mu)$ in (17.12) that changes, and the value of $N(\mu)$ of course also changes when $\mu\pi^2$ replaces $\mu$. Thus it is easy to see that for $D = 8, 12$,

$$Z(\mu\pi^2) = \begin{cases} Z(\mu)|N(\pi)|, & N(\pi) > 0, \\ Z(\mu)[|N(\pi)|+2], & N(\pi) < 0. \end{cases} \tag{26.2}$$

In the case where $D = 8$, $Z(\mu) = A_3(\mu)$ and (26.2) gives us a theorem on the comparative number of representations of $\mu$ and $\pi^2\mu$ by three squares. Such theorems would not be easy to prove directly.

For example, we might start with $A_3(1) = 6$ and a prime

$$p = 8K \pm 1, \tag{26.3}$$

so that in $R(2^{\frac{1}{2}})$

$$\pm p = \pi\pi'. \tag{26.4}$$

Then using some of the results of (19.4) and (26.2) we learn the equation, in $R(2^{\frac{1}{2}})$,

$$\pi^2 = \xi_1^2 + \xi_2^2 + \xi_3^2 \tag{26.5}$$

has $48K$ non-trivial solutions (the triviality $\pi^2 = \pi^2$ counting for six additional ones). If

$$p \equiv \pm 3 \pmod 8, \tag{26.6}$$

so that $p$ is prime in $R(2^{\frac{1}{2}})$, the equation

$$p^2 = \xi_1^2 + \xi_2^2 + \xi_3^2 \tag{26.7}$$

has $6(p^2 - 1)$ non-trivial solutions.

It would be irrelevant to our goal to attempt to categorize the ensuing wealth of non-analytic results.

In concluding, we might note that starting with hyperelliptic integrals and using Hermite's analysis, Humbert [19] discovered that $D = 5, 8$, and 12 all enjoy a favored role in theta-function theory, but he did not press the matter far enough to discover any number theoretic results of intrinsic interest.

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# POINTS MULTIPLES D'UNE APPLICATION ET PRODUIT CYCLIQUE REDUIT.*

par André Haefliger.

Le but essentiel de cette note est de déterminer la classe de cohomologie universelle modulo $p$ duale au cycle des points $p$-uples d'une application $f$ d'une variété $V$ dans une variété $M$, $p$ étant premier et les points $p$-uples étant considérés comme des points du produit cyclique de $V$. Cette classe peut aussi s'interprèter comme une obstruction à trouver dans la classe d'homotopie de $f$ une application sans point $p$-uple. Elle est en relation étroite avec la classe de plongement $\Phi_p$ de Wu (cf. [13]).

La méthode utilisée donne une détermination explicite de la cohomologie modulo $p$ du $p$-produit cyclique réduit $V^*{}_p$ d'une variété $V$. C'est dans la cohomologie de cet espace que se trouvent des obstructions au plongement de $V$ dans une variété $M$ (cf. [13] et [7]). Nous retrouvons les conditions données par Wu pour l'annulation des classes $\Phi_p$ lorsque $M$ est l'espace euclidien.

Je tiens à remercier vivement le Prof. N. E. Steenrod qui m'a communiqué ses résultats avant leur publication.

**1. Définitions.** Ce paragraphe rappelle essentiellement les notions introduites dans [5].

1.1. *Point p-uple de type $\pi$ d'une application.* Soit $f$ une application continue d'un espace $V$ dans un espace $M$. Un point $p$-uple de $f$ est une suite $x_1, \cdots, x_p$ de $p$ points distincts de $V$ tels que $f(x_1) = \cdots = f(x_p)$. Mais il est naturel d'identifier deux telles suites si l'une se déduit de l'autre par une permutation. Soit donc $\pi$ un groupe de permutations de $p$ objets; il agit sur le produit $V^p$ par permutation des facteurs et, par restriction, sur le sous-espace $V^p{}_0$ formé des suites de $p$ points distincts de $V$. Soit $V^0{}_\pi$ le quotient de $V^p{}_0$ par l'action de $\pi$: deux points $(x_1, \cdots, x_p)$ et $(x'_1, \cdots, x'_p)$ de $V^p{}_0$ définissent le même point $\{x_1, \cdots, x_p\}$ de $V^0{}_\pi$ si l'une des suites est obtenue à partir de l'autre par une permutation appartenant à $\pi$. Par définition, *un point p-uple de f de type $\pi$* est un point $\{x_1, \cdots, x_p\}$ de $V^0{}_\pi$ tel que $f(x_1) = \cdots = f(x_p)$.

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1.2. *Classe universelle des points p-uples.* Supposons que $M$ soit une variété de dimension $m$. Soit $E_\pi \to B_\pi$ un espace fibré universel de groupe structural $\pi$. Soit $M^p{}_\pi$ le fibré $E_\pi \times_\pi M^p$ associé à $E_\pi$ de fibre $M^p$ sur laquelle $\pi$ agit par permutation des facteurs. La reunion des diagonales des fibres forme un sous-fibré trivial $M_\pi = B_\pi \times M$: le sous-fibré diagonal.

On peut définir la classe $\Delta^M{}_\pi$ duale à $M_\pi$ dans $M^p{}_\pi$ de la manière suivante. Soit $E'_\pi \to B'_\pi$ un fibré universel de groupe $\pi$ pour une grande dimension $N$ et tel que $E'_\pi$ et $B'_\pi$ soient des variétés; alors le fibré $M'^p{}_\pi$ associé à $E'_\pi$ de fibre $M^p$ est une variété ainsi que le sous-fibré diagonal $M'_\pi$. Soit $\Delta'^M{}_\pi$ la classe duale à la classe d'homologie de $M'^p{}_\pi$ représentée par la sous-variété $M'_\pi$; c'est un élément de $H^{(p-1)m}(M'^p{}_\pi)$, les coefficients étant les entiers modulo 2 si $M$ est non orientable ou un faisceau d'entiers tordus si $M$ est orientée. Il existe une représentation définie à l'homotopie près de $M'^p{}_\pi$ dans $M^p{}_\pi$; elle induit un isomorphisme de $H^{(p-1)m}(M^p{}_\pi)$ sur $H^{(p-1)m}(M'^p{}_\pi)$ si $N$ est assez grand. Par définition $\Delta^M{}_\pi$ sera l'élément de $H^{(p-1)m}(M^p{}_\pi)$ correspondant à $\Delta'^M{}_\pi$ par cet isomorphisme.

La classe $\Delta^M{}_\pi$ (ou plus simplement $\Delta_\pi$) sera la *classe universelle des points p-uples de type* $\pi$ pour les applications continues dans $M$. Cette terminologie est justifiée dans les paragraphes qui suivent.

1.3. *La classe* $O^f{}_\pi$. L'espace $V^p{}_0$ (cf. 1.1) est un revêtement de $V^0{}_\pi$ et peut être considéré comme un fibré principal de groupe structural $\pi$. Soit alors comme plus haut $E$ le fibré associé à $V^p{}_0$ de fibre $M^p$ et soit $D$ le sous-fibré diagonal formé de la réunion des diagonales des fibres. Toute application continue $f$ de $V$ dans $M$ définit une section $f^0{}_\pi$ de $E$ et le point $x = \{x_1, \cdots, x_p\}$ de $V^0{}_\pi$ est un point $p$-uple de f si et seulement si $f^0{}_\pi(x)$ appartient à $D$.

Si $V$ est un espace assez raisonnable (par exemple un polyèdre), il existe une représentation $h$ de $E$ dans l'universel $M^p{}_\pi$ unique à l'homotopie près.

*Par définition la classe* $O^f{}_\pi$ *est égale à* $f^{0*}{}_\pi h^*(\Delta^M{}_\pi)$. Il est clair que $O^f{}_\pi$ ne dépend que la classe d'homotopie de $f$.

PROPOSITION. *Si $f$ est homotope à une application de $V$ dans $M$ sans point p-uple,alors* $O^f{}_\pi = 0$.

En effet, si $f$ est une application sans point $p$-uple, alors $hf^0{}_\pi$ applique $V^0{}_\pi$ dans $M^p{}_\pi - M_\pi$; d'autre part l'image de $\Delta^M{}_\pi$ par l'homomorphisme induit par l'injection de $M^p{}_\pi - M_\pi$ dans $M^p{}_\pi$ est nulle, car $\Delta^M{}_\pi$ est représentée par une cochaine de support $M_\pi$.

La classe $O^f{}_\pi$ représente dans un certain sens une première obstruction

à trouver dans la classe d'homotopie de $f$ une application sans point $p$-uple (cf. [7], [13]).

1.4. *Interprétation de la classe $O^f{}_\pi$ dans le case différentiable.* Soit $f$ une application différentiable d'une variété $V$ dans une variété $M$ présentant sous forme régulière les points $p$-uples, c'est à dire que pour tout point $p$-uple $(x_1, \cdots, x_p)$ de $f$ les images par $f$ des espaces tangents à $V$ en $x_1, \cdots, x_p$ sont en position générale dans l'espace tangent à $M$ en $y = f(x_1) = \cdots = f(x_p)$. Ceci équivaut à dire que $f^p\colon V^p \to M^p$ est transverse (cf. [11]) sur la diagonale de $M^p$ en tout point de $V^p{}_0 \subset V^p$, ou encore que la section $f^0{}_\pi$ du fibré $E$ est transverse sur le sous-fibré diagonal $D$. Dans ces conditions, *les points $p$-uples de type $\pi$ de $f$ forment une sous-variété dans $V^0{}_\pi$ représentant une classe d'homologie duale à $O^f{}_\pi$.* Cela résulte immédiatement de la définition de $O^f{}_\pi$ car $h^*(\Delta^M{}_\pi)$ est la classe duale à $D$ dans $E$.

On peut remarquer (cf. [5], I, 3) que l'on peut toujours approcher une application donnée par une application différentiable qui présente sous forme régulière les $p$-uples sur un ouvert de $V^0{}_\pi$ qui en est un rétract par déformation.

1.5. *Produit cyclique réduit $V^*{}_p$ et classes de plongement $\phi^f{}_p$.* Soit $p$ un nombre premier. Suivant la terminologie de Wu [13], appelons *$p$-produit cyclique réduit de $V$* le quotient $V^*{}_p$ de $V^p - V$ ($p$-produit de $V$ privé de sa diagonale $V$) par l'action du groupe $\pi$ des permutations cycliques des facteurs. Comme $p$ est premier, l'espace $V^p - V$ est un revêtement à $p$ feuillets de $V^*{}_p$ et peut être aussi considéré comme un espace fibré principal de groupe structural $\pi$. On peut donc construire comme plus haut (1.2 et 1.3) le fibré associé (de base $V^*{}_p$) de fibre $M^p$ et l'on a une représentation $h$ de $E$ dans l'universel $M^p{}_\pi$; toute application continue $f$ de $V$ dans $M$ définit une section $f_p$ de $E$.

*Par définition la classe $\phi^f{}_p \in H^{(p-1)m}(V^*{}_p)$ est égale à $f^*{}_p h^*(\Delta^M{}_\pi)$.*

On remarquera que $O^f{}_\pi$ est l'image de $\phi^f{}_p$ par l'homomorphisme induit par l'injection du sous-espace $V^0{}_\pi$ dans $V^*{}_p$. Naturellement pour $p = 2$, $O^f{}_\pi = \phi^f{}_p$.

PROPOSITION. *Si $f$ est homotope à un plongement de $V$ dans $M$, alors* $\phi^f{}_p = 0$.

La démonstration est la même que celle de la proposition précédente.

Les classes $\phi^f{}_p$ généralisent au signe près les classes de plongement $\phi^{(p-1)m}{}_p$ définies par Wu lorsque $M$ est l'espace euclidien $R^m$ (cf. [13], II).

On pourrait aussi définir, suivant Wu (cf. [13], II), les classes d'im-

mersion $\psi^f{}_p$ en prenant l'image de $\phi^f{}_p$ dans la limite inductive des cohomologies des voisinages de la diagonale dans le $p$-produit cyclique réduit de $V$. Si $f$ est homotope à une immersion (c. à d. une application localement biunivoque), alors $\psi^f{}_p = 0$.

**2. Résultats de Steenrod.** Dans tout ce qui suit, $p$ est un nombre premier et $\pi$ est le groupe des permutations cycliques de $p$ objets; $T$ est un générateur de $\pi$. Tous les groupes de cohomologie considérés sont à coefficients les entiers modulo $p$.

2.1. *L'anneau de cohomologie du classifiant* $B_\pi$ pour le groupe $\pi$ sera noté $H^*(\pi)$ (cf. [3], Chap. XII).

Pour $p = 2$, $H^*(\pi) = P(\mu)$, où $P(\mu)$ est l'anneau des polynomes sur $Z_2$ dans la variable $\mu \in H^1(\pi)$.

Pour $p > 2$, $H^*(\pi) = P(\mu) \otimes E(\nu)$, où $P(\mu)$ est l'anneau des polynomes sur $Z_p$ dans la variable $\mu \in H^2(\pi)$ et $E(\nu)$ l'algèbre extérieure engendrée par un élément $\nu \in H^1(\pi)$.

La cohomologie de tout espace fibré $q\colon E \to B$ à groupe structural $\pi$ est une $H^*(\pi)$-algèbre: soit $h$ l'application (définie à une homotopie près) de $B$ dans $B_\pi$ qui définit la classe de $E$; si $x \in H^*(E)$ et $\alpha \in H^*(\pi)$, par définition $\alpha x = q^*h^*(\alpha) \cup x$.

2.2. *Calcul de* $H^*(M^p{}_\pi)$. Soit $M$ un complexe fini et soit $M^p{}_\pi$ le fibré associé à $E_\pi$ de fibre $M^p$ sur laquelle $\pi$ agit par permutations cycliques des facteurs (cf. 1.2).

THÉORÈME (Steenrod). *Il existe un isomorphisme naturel de* $H^*(\pi)$-*algèbre* $\phi\colon H^*(M^p{}_\pi) \to H^*(\pi, H^*(M)^p)$.

*La projection naturelle de* $H^*(\pi, H^*(M)^p) = \sum_{r \geq 0} H^r(\pi, H^*(M)^p)$ *sur* $H^0(\pi, H^*(M)^p)$ *identifié au sous-groupe des éléments invariants de* $H^*(M)^p = H^*(M^p)$ *correspond à l'homomorphisme* $r^*\colon H^*(M^p{}_\pi) \to H^*(M^p)$ *induit par l'injection de la fibre* $M^p{}_\pi$.

Dans cet énoncé, $H^*(M)^p$ est consideré comme un $\pi$-module (c. à d. un module sur l'anneau $Z(\pi)$ de $\pi$), $\pi$ agissant par permutation cyclique des facteurs du produit tensoriel $H^*(M)^p$ avec le changement de signe usuel (cf. [3]).

2.3. Avant d'esquisser la démonstration, explicitons la structure de $H^*(\pi, H^*(M)^p)$

Soit $D$ le sous-groupe de $H^*(M)^p$ formé des combinaisons linéaires des éléments diagonaux $(x)^p = x \otimes x \otimes \cdots \otimes x$, où $x \in H^*(M)$; le groupe $\pi$ laisse fixe les éléments de $D$. *Comme $\pi$-module, $H^*(M)^p$ est somme directe de $D$ et d'un sous-module libre.* Soit en effet $(a_i)$ une base de $H^*(M)$; les éléments $a_{i_1} \otimes \cdots \otimes a_{i_p}$ forment une base de $H^*(M)^p$. Les éléments $a_i \otimes \cdots \otimes a_i$ forment une base de $D$ et les autres éléments de la base engendrent un sous-module $\pi$-libre que l'on peut écrire sous la forme $Z_p(\pi) \otimes S$, où $S$ est l'espace vectoriel sur $Z_p$ engendré par une $\pi$-base.

Ainsi $H^*(M)^p = D + Z_p(\pi) \otimes S$ et donc

$$H^*(\pi, H^*(M)^p) = H^*(\pi) \otimes D + N,$$

où $N$ désigne l'image de $H^*(M)^p$ par l'opérateur norme $N: 1 + T + \cdots + T^{p-1}$ (cf. [3], Chap. XII).

Le $H^*(\pi)$-module $H^*(\pi, H^*(M)^p)$ est donc engendré par le sous-groupe $D + N$ des éléments de $\pi$-degré 0 (un élément de $H^r(\pi, H^*(M)^p)$ est de $\pi$-degré $r$); de plus $H^*(\pi)$ agit trivialement sur les normes $N$ (cf. [3], Chap. XII).

La structure multiplicative est déterminée en remarquant que les éléments de $\pi$-degré 0 forment un sous-anneau isomorphe à celui des éléments invariants de $H^*(M)^p$ et que $H^*(\pi) \otimes D$ est une sous-algèbre produit extérieure des anneaux $H^*(\pi)$ et $D$. On remarquera enfin que l'application de $H^*(M)$ sur $D$ faisant correspondre à un élément $x$ sa $p$-ème puissance tensorielsle $(x)^p$ est un homomorphisme d'anneaux *modulo $N$*.

2.4. La démonstration du théorème est en gros la suivante (pour plus de détails, voir [9]).

Soit $W$ le complexe des chaines de $B$ (résolution acyclique libre du $\pi$-module $Z$) et soient $C_*(M)$ (resp. $C^*(M)$) le complexe des chaines (resp. des cochaines) à coefficients $Z_p$ de $M$. Alors $H^*(M^p{}_\pi)$ est la cohomologie du complexe double $\mathrm{Hom}(W \otimes_\pi C^*(M)^p, Z_p)$ qui s'identifie canoniquement au complexe double $\mathrm{Hom}_\pi(W, C^*(M)^p)$. Comme les coefficients sont dans un corps, il existe une équivalence naturelle (à l'homotopie près) entre le complexe de chaines $C^*(M)$ et $H^*(M)$ considéré comme un complexe de chaines avec opérateur bord zéro; il en résulte également que les $\pi$-complexes de chaines $C^*(M)^p$ et $H^*(M)^p$ sont équivalents. Donc $\mathrm{Hom}_\pi(W, C^*(M)^p)$ est équivalent (comme $\pi$-complexe) au complexe simple $\mathrm{Hom}_\pi(W, H^*(M)^p)$ dont la cohomologie est par définition $H^*(\pi, H^*(M)^p)$. Il existe donc un isomorphisme naturel $\phi: H^*(M^p{}_\pi) \to H^*(\pi, H^*(M)^p)$ de $H^*(\pi)$-modules.

La deuxième partie de l'énoncé résulte immédiatement de la définition

de $\phi$. Ceci montre que la restriction de $\phi$ aux éléments de $\pi$-degré 0 est un homomorphisme multiplicatif; comme $H^*(M)^p$ est engendré comme $H^*(\pi)$-module par ses éléments de $\pi$-degré 0 (cf. 2.3), il en résulte que $\phi$ est aussi un isomorphisme d'anneaux.

2.5. *L'homomorphisme* $i^*: H^*(M^p{}_\pi) \to H^*(M_\pi)$. Soit $M_\pi$ le sous-fibré diagonal de $M^p{}_\pi$ (cf. 1.2) ; il est isomorphe au produit $B_\pi \times M$.

Avec l'identification du Théorème 2.2 et l'identification de $H^*(M_\pi)$ avec $H^*(\pi) \otimes H^*(M)$, l'homomorphisme $i^*$ induit par l'injection de $M_\pi$ dans $M^p{}_\pi$ sera un homomorphisme de $H^*(\pi)$-algèbre:

$$i^*: H^*(\pi, H^*(M)^p) \to H^*(\pi) \otimes H^*(M).$$

Il suffira de connaître $i^*$ sur le sous-groupe $D + N = H^0(\pi, H^*(M)^p)$ qui engendre $H^*(\pi, H^*(M)^p)$. Comme $H^*(\pi)$ agit trivialement sur $N$ et que $H^*(\pi) \otimes H^*(M)$ est un $H^*(\pi)$-module libre, on a $i^*(N) = 0$. La valeur de $i^*$ sur $D$ est donnée par le

THÉORÈME (Steenrod). *Soit $x$ un élément de $H^q(M)$ et $(x)^p = x \otimes \cdots \otimes x$. On a* $p = 2$, $i^*(x)^p = \sum_0^q \mu^{q-i} Sq^i x$,

$$p > 2,\ i^*(x)^p = \sigma \sum_{i=0}^{[q/2]} (-1)^i \mu^{h(q-2i)} \mathcal{P}^i x - \sigma \sum_{i>0}^{[q/2]} (-1)^i \mu^{h(q-2i)-1} \nu \beta \mathcal{P}^i x,$$

*où les $\mathcal{P}^i$ (resp. les $Sq^i$) sont les $p$-èmes puissances cycliques r-duites (resp. les carrés) de Steenrod (cf.* [8]); *$\beta$ est l'homomorphisme $H^k(M, Z_p) \to H^{k+1}(M, Z_p)$ de Bockstein; $h = (p-1)/2$ et $\sigma = (-1)^{q/2}$ ou $(-1)^{(q-1)/2} h!$ suivant que $q$ est pair our impair.*

Pour la démonstration, cf. [9]. On remarquera que $i^*$ est injectif sur le sous-module $H^*(\pi) \otimes D$.

## 3. Détermination de la classe universelle des points $p$-uples.

3.1. Soit $M$ une variété connexe de dimension $n$, compacte, avec un bord $B$. Nous supposerons $M$ orientée si $p > 2$. Pour tout nombre premier $p$ et tout entier $j$ positif, WU a défini (cf. [12]) les classes $U^j{}_{(p)}$ appartenant à $H^{2j(p-1)}(M, Z_p)$ pour $p > 2$ ou à $H^j(M, Z_2)$ pour $p = 2$ par les relations:

$$< Sq^j \alpha, M > = < U^j{}_{(2)} \cup \alpha, M >, \qquad p = 2,$$

$$< \mathcal{P}^j \alpha, M > = < U^j{}_{(p)} \cup \alpha, M >, \qquad p > 2,$$

pour tout élément $\alpha$ appartenant à $H^*(M \bmod B, Z_p)$ ; le symbole $< \gamma, M >$

désigne la valeur (indice de Kronecker) de $\gamma \in H^r(M, Z_p)$ sur le générateur de $H_m(M \bmod B, Z_p)$ défini par l'orientation de $M$. Remarquons que $< \gamma, M >$ $= 0$ si $r \neq m$.

Pour rester dans le cadre de la théorie précédente, nous supposerons que $M$ est un complexe.

3.2. Théorème. *La classe $\Delta_\pi$ modulo $p$ duale au sous-fibré diagonal $M$ de $M^p$ est*

$$\Delta_\pi = \sum_{j=0}^{\leq m/2} \mu^{m-2j}(U^j{}_{(2)})^2 + \delta_2, \qquad p = 2,$$

$$\Delta_\pi = \lambda \sum_{j=0}^{\leq m/2p} (-1)^j \mu^{h(m-2jp)} (U^j{}_{(p)})^p + \delta_p \qquad p > 2,$$

*où $\delta_p$ est la classe duale dans $M^p$ à la classe d'homologie représentée par la diagonale, $h = (p-1)/2$ et $\lambda = (-1)^{m/2}$ ou $(-1)^{(m-1)/2} h!$ suivant que $m$ est pair ou impair.*

Pour exprimer $\Delta_\pi$, on a identifié $H^*(M^p{}_\pi)$ à

$$H^*(\pi, H^*(M)^p) = \sum_{k>0} H^k(\pi) \otimes D + (D + N)$$

(cf. 2.3).

3.3. Démonstration. Prenons pour $E_\pi$ le complexe standard qui a $p$ cellules dans chaque dimension $i \geqq 0$: $e_i, Te_i, \cdots, T^{p-1}e_i$, l'operateur bord étant défini par $\partial e_{2n} = (1 + T + \cdots + T^{p-1}) e_{2n-1}$ et $\partial e_{2n+1} = (1 - T) e_{2n}$.

Remplaçons le fibré $E_\pi \to B_\pi$ par le fibré $E'_\pi \to B'_\pi$ obtenu en enlevant de $E_\pi$ toutes les cellules de dimension plus grandes que $(p-1)N - 1$, où $N > m$. Nous obtenons ainsi une décomposition cellulaire de la sphère $S^{(p-1)N-1}$ et $B'_\pi$ est un espace lenticulaire de la même dimension. Soit $M'^p{}_\pi$ la restriction du fibré $M^p{}_\pi$ à $B'_\pi$.

Le même raisonnement que dans 2.4 montre qu'il existe un isomorphisme naturel de $H^*(\pi)$-module $\phi'$ tel que le diagramme suivant soit commutatif:

$$\begin{tikzcd}
H^*(M'^p{}_\pi) \arrow[r, "\phi'"] & H^*(B'_\pi, H^*(M)^p) \\
H^*(M^p{}_\pi) \arrow[r, "\phi"] \arrow[u, "j^*"] & H^*(B_\pi, H^*(M)^p) \arrow[u, "j^*"']
\end{tikzcd}$$

où les homomorphismes verticaux $j^*$ sont induits par l'inclusion de $M'^p{}_\pi$ dans $M^p{}_\pi$ et $B'_\pi$ dans $B_\pi$ resp.; $j^*$ est bijectif pour $r < (p-1)N - 1$, injectif pour $r = (p-1)N - 1$ et zéro pour $r > (p-1)N - 1$, $r$ désignant le $\pi$-degré

(cf. 2.3). De plus $\phi'$ est aussi multiplicatif (tout au moins sur l'image de $j^*$ puisque $\phi$ et $j^*$ sont des homomorphismes d'anneaux).

Choisissons un générateur $S$ de $H_{(p-1)N-1}(B'_\pi, Z_p)$ tel que $<\mu^{N-1}, S> = 1$ si $p=2$ et $<\mu^{hN-1}\nu, S> = 1$ si $p>2$; soit $M \in H_m(M \operatorname{mod} B, Z_p)$ le générateur défini par l'orientation de $M$. Alors les éléments $S \otimes M$ et $S \otimes M^p$ sont des classes d'homologie qui définissent des orientations modulo $p$ de $M'_\pi$ et $M'^p_\pi$ respectivement.

Soit $\Delta_\pi$ la classe duale à $M'_\pi$ relativement à ces orientations. Pour tout $x \in H^{(p-1)N-1+m}(M'^p_\pi)$, on a

3.4. $$< i^*(x), S \otimes M > = < x \cup \Delta_\pi, S \otimes M^p >$$

car par définition $\Delta_\pi \cap (S \otimes M^p) = i_*(S \otimes M)$ et $< x \cup \Delta_\pi, S \otimes M^p > = < x, \Delta_\pi \cap (S \otimes M^p) > = < x, i_*(S \otimes M) > = < i^*(x), S \otimes M >$ (cf. [10]).

La formule 3.4 permet de calculer $\Delta_\pi$ modulo les normes $N$ en appliquant mécaniquement 2.3 et 2.5. Montrons-le dans le cas $p=2$. Posons $\Delta_\pi = \sum \mu^{m-2j}(V^j)^2$ modulo $N$, où $V^j \in H^j(M)$. Soit $\alpha \in H^{m-i}(M \operatorname{mod} B)$ et soit $x = \mu^{N-m+2i-1}(\alpha)^2$. D'après 2.5, on a $i^*(x) = \sum \mu^{N+i-j-1} Sq^j \alpha = \mu^{N-1} Sq^i \alpha$ pour raison de dimensions. De même, d'après 2.3,

$$x \cup \Delta_\pi = \sum \mu^{N+2i-2j-1}(V^j\alpha)^2 = \mu^{N-1}(V^i\alpha)^2.$$

Donc

$$< i^*(x), S \otimes M > = < Sq^i\alpha, M > \text{ et } < x \cup \Delta_\pi, S \otimes M^2 > = < V^i\alpha, M >$$

d'où $V^i = U^i_{(2)}$.

Un calcul analogue donne dans le cas $p>2$

$$\Delta_\pi = \lambda \sum_{0 \leq j \leq m/2p} (-1)^j \mu^{h(m-2jp)} (U^j_{(p)})^p \text{ modulo } N.$$

Pour déterminer $\Delta_\pi$, il suffit de connaître so composante de $\pi$-degré 0 (cf. 2.3). Remarquons pour cela que l'injection $r$ de la fibre $M^p$ dans $M'^p_\pi$ est transverse sur la sous-variété $M'_\pi$ et que $r^{-1}(M'_\pi)$ est la diagonale de $M^p$. Donc $r^*(\Delta_\pi)$ est la classe duale $\delta_p$ à la diagonale dans $M^p$ (cf. [11]); comme $r^*$ est injectif sur les éléments de $\pi$-degré 0 et les identifie aux éléments invariants de $H^*(M^p)$ (cf. 2.2 et 2.3), le théorème est démontré.

3.5. *Classes caractéristiques de WU.* Posons

$$W^i = \sum_j Sq^{i-j} U^j_{(2)}$$

et

$$Q^i_{(p)} = \sum_j \mathcal{P}^{i-j} U^j_{(p)}, \qquad p > 2.$$

Les classes $W^i$ sont les classes de Stiefel-Whitney de $M$ et les classes $Q^i_{(p)} \in H^{2(p-1)i}(M, Z_p)$ (notées aussi $Q^i$) sont les classes caractéristiques de Wu. Si $M$ est une variété différentiable, ces classes sont des polynomes dans les classes de Pontryagin de $M$ réduites modulo $p$ (cf. [6]).

Considérons la classes $i^*(\Delta_\pi)$ restriction de $\Delta_\pi$ à $M_\pi$; elle peut s'interprèter comme la classe duale à la self-intersection de la "sous-variété" $M_\pi$ de $M^p{}_\pi$; dans le cas différentiable, $i^*(\Delta_\pi)$ est la classe d'Euler du fibré normal à $M_\pi$ dans $M^p{}_\pi$ (c'est à dire la réunion des fibrés normaux aux diagonales des fibres ou encore la limite du fibré normal à $M'_\pi$ dans $M'^p{}_\pi$ lorsque $N$ tend vers l'infini).

3.6. Corollaire.

$$i^*(\Delta_\pi) = \sum_i \mu^{m-i} W^i, \qquad p = 2$$

$$i^*(\Delta_\pi) = \lambda \sum_k (-1)^k \mu^{h(m-2k)} Q^k{}_{(p)} - \lambda \sum_k (-1)^k \mu^{h(m-2k)-1} \nu \beta Q^k{}_{(p)}, \qquad p > 2.$$

Ces expressions se calculent immédiatement en appliquant 2.5 et 3.2. Lorsque $M$ est une variété différentiable, alors $\beta Q^k{}_{(p)} = 0$ car les $Q^k$ sont la réduction modulo $p$ de classes entières (polynomes dans les classes de Pontryagin) (cf. [11], p. 63).

Les classes $i^*(\Delta_\pi)$ s'interprètent aussi comme classes universelles d'immersion dans $M$ (cf. 1.5, 5.4).

3.7. *Expression de* $\phi^f{}_p$. Remarquons tout d'abord que $V^p{}_\pi - V_\pi$ a le même type d'homotopie que le $p$-produit cyclique réduit $V^*{}_p$ de $V$. En effet $V^p{}_\pi - V_\pi = E_\pi \times_\pi (V^p - V)$ est fibré sur $V^*{}_p$ avec une fibre contractile $E_\pi$. La projection $q$ de cette fibration induit un isomorphisme naturel de $H^*(V^*{}_p)$ sur $H^*(V^p{}_\pi - V_\pi)$; avec cette identification, $j^*: H^*(V^p{}_\pi) \to H^*(V^*{}_p)$ désignera l'homomorphisme induit par l'injection $j$ de $V^p{}_\pi - V_\pi$ dans $V^p{}_\pi$.

Toute application continue $f$ de $V$ dans $M$ induit une application continue $\phi^p{}_\pi = 1 \times_\pi f^p$ de $V^p{}_\pi = E_\pi \times_\pi V^p$ dans $M^p = E_\pi \times_\pi V^p$. Avec les identifications de 2.2, $\phi^{p*}{}_\pi$ n'est autre que l'homomorphisme de $H^*(\pi, H^*(M)^p)$ dans $H^*(\pi, H^*(V)^p)$ induit par l'homomorphisme $f^{*p}: H^*(M)^p \to H^*(V)^p$.

On vérifie aisément (notations de 1.5) que $hf_\pi q$ est homotope à $\phi^p{}_\pi j$. Donc $\phi^f{}_p = j^*[\phi^{p*}{}_\pi \Delta_\pi]$.

Pour $p = 2$, on aura par exemple

$$\phi^f{}_2 = j^*[\sum \mu^{m-i}(f^* U^i)^2 + f^{2*}(\delta_2)].$$

L'homomorphisme $j^*$ sera explicité au paragraphe suivant dans le cas où $V$ est une variété.

## 4. Cohomologie modulo $p$ du $p$-produit cyclique réduit d'une variété.

4.1. *Suites exactes associées à une sous-variété.* Soit $V$ une sous-variété fermée de codimension $q$ d'une variété paracompacte $M$. Dan ce paragraphe l'homologie ou la cohomologie sont à coefficients entiers ou modulo 2 suivant que les variétés sont orientées ou non, et la famille des supports est celle de tous les fermés. En utilisant la cohomologie définie par A. Borel dans [1] (cf. aussi [2]), $U$ étant un voisinage ouvert de $V$, on a la suite exacte suivante:

$$\cdots \to H_k(V) \to H_k(U) \to H_k(U - V) \to H_{k-1}(V) \to \cdots$$

où le premier homomorphisme est induit par l'injection $i$ de $V$ dans $U$ et le second par la restriction à $U - V$ des chaines de $U$. Si $U'$ est un voisinage ouvert de $V$ contenu dans $U$, cette suite exacte s'envoie par restriction dans la suite exacte analogue pour $U'$. En posant d'une part $U = M$ et en passant d'autre part à la limite directe suivant l'ordonné filtrant des voisinages de $V$, on a le diagramme commutatif et exact suivant:

$$\begin{tikzcd}
\to H_k(V) \arrow[r] \arrow[d] & H_k(M) \arrow[r] \arrow[d] & H_k(M - V) \to \arrow[d] \\
\to H_k(V) \arrow[r] & \text{lim. dir.}\, H_k(U) \arrow[r] & \text{lim. dir.}\, H_k(U - V) \to
\end{tikzcd}$$

Passons à la cohomologie par dualité de Poincaré (cf. [2]). L'injection de $V$ dans $U$ induit un isomorphisme de lim. dir. $H^r(U)$ sur $H^r(V)$ (cf. [4]). Avec cette identification et en notant $H^r(M \setminus V)$ la limite directe de $H^r(U - V)$ lorsque $U$ parcourt les voisinages ouverts de $V$ dans $M$, on a

4.2. PROPOSITION. *Le diagramme suivant est commutatif.*

$$\begin{tikzcd}
{} \arrow[dr] & & H^r(M) \arrow[r, "j^*"] \arrow[d] & H^r(M - V) \arrow[d] \arrow[dr] & & {} \\
& H^{r-q}(V) \arrow[ur, "\phi"] \arrow[dr, "\phi_0"'] & & & H^{r-q+1}(V) \arrow[ur] \arrow[dr] & \\
{} \arrow[ur] & & H^r(V) \arrow[r, "j^*_0"] & H^r(M \setminus V) \arrow[ur] & & {}
\end{tikzcd}$$

*Les lignes horizontales sont exactes, $\phi$ est l'homomorphisme de Gysin déterminé par l'injection $i$ de $V$ dans $M$, $j^*$ est induit par l'injection de $M - V$ dans $M$. L'homomorphisme $\phi_0$ est déterminé par le cup produit par la classe $i^*(V^*)$, où $V^*$ est la classe duale à $V$ dans $M$.*

Il reste à vérifier ce dernier point. Comme lim. dir $H^*(U) = H^*(V)$,

pour tout $\alpha \in H^*(V)$, il existe un voisinage ouvert $U$ de $V$ et un élément $\beta \in H^*(U)$ tel que $i^*_U(\beta) = \alpha$, où $i_U$ est l'injection de $V$ dans $U$; si $\phi_U$ désigne l'homomorphisme de Gysin déterminé par $i_U$, on a $\phi_0(\alpha) = i^*_U \phi_U(\alpha) = i^*_U \phi_U(i^*_U \beta) = i^*_U(\beta \cup V^*) = \alpha \cup i^*(V^*)$.

Remarquons que si $V$ est une sous-variété différentiable de $M$, alors $H^*(M \setminus V)$ s'identifie à la cohomologie du bord (espace fibré en sphères) d'un voisinage tubulaire de $V$ et la deuxième suite exacte n'est autre que la suite exacte de Gysin.

On peut compléter le diagramme ci-dessus par des suites exactes verticales en faisant intervenir le groupe $H^*(M \bmod V)$.

4.3. *Application à la "sous-variété" $M_\pi$ de $M^p{}_\pi$.* Comme dans 3.3, réalisons $E_\pi$ (resp. $B_\pi$) par une sphère (resp. un espace lenticulaire) de grande dimension $(p-1)N-1$. Ecrivons les suites exactes de 4.2 associées à la sous-variété $M'_\pi$ de $M'^p{}_\pi$ et passons à la limite en faisant tendre $N$ vers l'infini. Nous obtenons les suites exactes:

$$\begin{tikzcd}
{} \arrow[dr] & & H^r(M^p{}_\pi) \arrow[r, "j^*"] \arrow[dd, "i^*"] & H^r(M^p{}_\pi - M_\pi) \arrow[dd] \arrow[dr] & & {} \\
& H^{r-(p-1)m}(M_\pi) \arrow[ur, "\phi"] \arrow[dr, "\phi_0"'] & & & H^{r-(p-1)m+1}(M_\pi) \arrow[ur] \arrow[dr] & \\
{} \arrow[ur] & & H^r(M_\pi) \arrow[r, "j^*_0"'] & H^r(M^p{}_\pi \setminus M_\pi) \arrow[ur] & & {}
\end{tikzcd}$$

Nous avons vu dans 3.7 que $M^p{}_\pi - M_\pi$ a même type d'homotopie que le $p$-produit cyclique réduit $M^*_p$ de $M$. Le même raisonnement montre que $H^*(M^p{}_\pi \setminus M_\pi)$ est canoniquement isomorphe à $H^*((M^p/\pi)\setminus M)$, où $M^p/\pi$ est le $p$-produit cyclique de $M$ (quotient de $M^p$ par l'action de $\pi$) et où $M$ est identifié à la diagonale de $M^p/\pi$.

D'autre part $\phi_0$ est donné par le cup produit par la classe $i^*(\Delta_\pi)$. Or le terme de $\pi$-degré maximum dans $i^*(\Delta_\pi)$ est $\mu^{mh}$ (cf. 3.6); il en résulte que si $a \in H^*(M_\pi)$ est $\neq 0$, alors $a \cup i^*(\Delta_\pi)$ est $\neq 0$, donc que $\phi_0$ est injectif. En vertu de la commutativité du diagramme, *$\phi$ est aussi injectif.*

Désignons par $r^*$ (resp. $r^*_0$) les homomorphismes induits sur la cohomologie par l'injection de la fibre $M^p$ (resp. $M$) dans $M^p{}_\pi$ (resp. $M_\pi$) par $\psi$ l'homomorphisme de Gysin déterminé par l'injection de la diagonale $M$ dans $M^p$.

4.4. Compte tenu de ce qui précède, on a le diagramme commutatif et exact horizontalement:

$$
\begin{tikzcd}
0 \arrow[r] & H^{r-(p-1)m}(M) \arrow[r, "\psi"] & H^r(M^p) & & \\
& & H^r(M^p{}_\pi) \arrow[u, "r^*"'] \arrow[r, "j^*"] \arrow[dd, "i^*"] & H^r(M^*{}_p) \arrow[dd] \arrow[dr] & \\
0 \arrow[r] & H^{r-(p-1)m}(B_\pi \times M) \arrow[uu, "r^*{}_0"] \arrow[ur, "\phi"] \arrow[dr, "\phi_0"'] & & & 0 \\
& & H^r(B_\pi \times M) \arrow[r, "j^*{}_0"] & H^r((M^p/\pi)\backslash M) \arrow[ur] &
\end{tikzcd}
$$

4.5. THÉORÈME. *L'homomorphisme* $j^*: H^r(M^p{}_\pi) \to H^r(M^*{}_p)$ *est surjectif. L'image par* $j^*$ *d'un élément* $\alpha \in H^*(M^p{}_\pi)$ *est nulle si et seulement si*

a) *il existe un élément* $\beta \in H^*(B_\pi \times M)$ *tel que* $i^*(\alpha) = \beta \cup i^*(\Delta_\pi)$ ($\beta$ *est alors unique*) *et*

b) $\psi r^*{}_0(\beta) = r^*(\alpha)$.

Les conditions a) et b) sont évidemment nécessaires d'après la commutativité de 4.4 et le fait que $\phi_0$ est donné par le cup produit par $i^*(\Delta_\pi)$.

Réciproquement, supposons qu'il existe $\beta$ tel que $i^*(\alpha) = \beta \cup i^*(\Delta_\pi)$; en vertu de 4.4, $\phi(\beta) - \alpha$ appartient au noyau de $i^*$; or ce noyau est contenu dans le sous-groupe des éléments de $\pi$-degré 0 de $H^*(M^p{}_\pi) = H^*(\pi, H^*(M)^p)$ (cf. 2.5); comme $r^*$ est injectif sur ce sous-groupe (cf. 2.2), la condition $r^*(\phi(\beta) - \alpha) = 0$, équivalente à b), entraine $\alpha = \phi(\beta)$, donc $j^*(\alpha) = 0$.

Ce théorème permet le calcul explicite de $H^*(M^*{}_p)$ compte tenu des identifications $H^*(M^p{}_\pi) = H^*(\pi, H^*(M)^p)$, $H^*(B_\pi \times M) = H^*(\pi) \otimes H^*(M)$, des expressions explicites de $i^*$ (cf. 2.5), de $r^*$ (2.2) et de $i^*(\Delta_\pi)$ (cf. 3.6).

Quant à $\psi$, il est déterminé par la formule $\psi(\gamma) = (\gamma \otimes 1 \cdots \otimes 1) \cup \delta_p$, où $\delta_p$ est la classe duale à la diagonale $M$ dans $M^p$ (on a identifié $H^*(M^p)$ à $H^*(M)^p$); en effet, si $i_0$ est l'injection de la diagonale $M$ dans $M^p$, $\gamma = i^*{}_0(\psi \otimes 1 \cdots \otimes 1)$ et la formule résulte de le propriété multiplicative de $\psi$.

## 5. Conditions pour l'annulation des classes de plongement $\varphi^f{}_p$.

5.1. *Classes caractéristiques normales d'une application.* Soit $f$ une application continue d'une variété $V$ de dimension $n$ dans une variété $M$ de dimension $m$. Soient $W = \sum W^i$ (resp. $W' = \sum W'^i$) la classe totale de Stiefel-Whitney de $V$ (resp. $M$) et $Q = \sum Q^i$ (resp. $Q' = \sum Q'^i$) la classe caractéristique (totale) modulo $p$ de Wu de $V$ (resp. $M$) (cf. 3.5).

*La classe totale de Stiefel-Whitney normale de* $f$ est la classe $\bar{W}_f = \sum \bar{W}^i{}_f = \bar{W} \cup f^*(W')$, où $\bar{W}$ est défini par $\bar{W} \cup W = 1$.

*La classe caractéristique modulo p de Wu normale de f est la classe* $\bar{Q}_f = \sum \bar{Q}^i{}_f = \bar{Q} \cup f^*(Q')$, où $\bar{Q}$ est défini par $\bar{Q} \cup Q = 1$.

Désignons par $\delta_p$ (resp. $\delta'_p$) la classe duale à la diagonale de $V^p$ (resp. $M^p$).

5.2. Théorème. *La condition* $\phi^f{}_p = 0$ *est équivalente aux conditions*

a) $\bar{W}^k{}_f = 0$, $k > m - n$ pour $p = 2$,

$\bar{Q}^k{}_f = 0$, $k > (m-n)/2$ pour $p > 2$

et

b) $(\bar{W}^{m-n}{}_f \otimes 1) \cup \delta_2 = f^{2*}(\delta'_2)$ pour $p = 2$,

$\sigma(\bar{Q}^{(m-n)/2}{}_f \otimes 1 \cdots \otimes 1) \cup \delta_p = f^{p*}(\delta'_p)$ pour $p > 2$

*où* $\sigma = 1$ *ou* $(-1)^{(p+1)/2}$ *suivant que m et n sont tous deux pairs ou impair.*

5.3. *Démonstration.* Reprenons les notations de 3.7, 4.4 et 4.5. La restriction de $\phi^p{}_\pi$ à $V_\pi$ est une application $\phi_\pi$ de $V_\pi$ dans $M_\pi$. On a vu (3.7) que $\phi^f{}_p = j^*\phi^{p*}{}_\pi\Delta^M{}_\pi$. D'après 4.5, $\phi^f{}_p = 0$ si et seulement si a)′ il existe $\beta \in H^*(B_\pi \times V)$ tel que $i^*(\phi^{p*}{}_\pi\Delta^M{}_\pi) = \beta \cup i^*(\Delta^V{}_\pi)$ et si b)′ $\psi r^*{}_0(\beta) = r^*(\phi^{p*}{}_\pi\Delta^M{}_\pi)$. Ces conditions a)′ et b)′ sont équivalentes respectivement aux conditions a) et b) de 5.2. Montrons le pour $p = 2$.

En effet, $i^*\phi^{p*}{}_\pi(\Delta^M{}_\pi) = \phi^*{}_\pi i^*(\Delta^M{}_\pi) = \sum \mu^{m-j} f^*(W'^j)$. La condition a)′ s'exprime sous la forme

a)′ $$\sum \mu^{m-j} f^*(W'^j) = \beta \cup \sum \mu^{n-i} W^i.$$

Multiplions les deux membres par $\sum \mu^{n-j}\bar{W}^j$. On obtient la condition $\sum \mu^{m+n-k}\bar{W}^k{}_f = \mu^{2n}\beta$ qui équivaut à $\bar{W}^k{}_f = 0$ pour $k > m - n$.

On voit aussi que $r^*{}_0(\beta) =$ composante de $\pi$-degré 0 de $\beta = \bar{W}^{m-n}{}_f$. Dans le cas $p > 2$, on a $r^*{}_0(\beta) = \sigma\bar{Q}_f{}^{(m-n)/2}$. La condition b)′ est bien équivalente à b) d'après la fin de 4.5.

5.4. *Les classes d'immersion* $\psi^f{}_p$. Il suit de cette démonstration que $\psi^f{}_p = 0$ est équivalent aux conditions a) de 5.2. En effet $\psi^f{}_p = j^*{}_0\phi^*{}_\pi i^*(\Delta^M{}_\pi)$.

5.5. Corollaire (Wu). *Si f est une application de V dans l'espace euclidien* $R^m$, *la condition* $\phi^f{}_p = 0$ *est équivalente à*

$\bar{W}^k = 0$ pour $k \geqq m - n$ dans le cas $p = 2$

$\bar{Q}^k = 0$ pour $k \geqq (m-n)/2$ dans le cas $p > 2$.

5.6. *Le cas différentiable.* On sait alors (cf. [6]) que la classe normale

$\bar{Q}^k{}_f$ de $f$ modulo $p$ est un polynome dans les classes de Pontryagin normales $\bar{p}^i{}_f \in H^{4i}(V)$ de $f$ réduites modulo p. Plus précisément, si $\bar{p}^i{}_f$ est formellement la $i$-ème fonction symétrique élémentaire $\sigma_i(x^2{}_1, \cdots, x^2{}_n)$, alors $\bar{Q}^k{}_f$ est la $k$-ème fonction symétrique élémentaire $\sigma_k(x^{p-1}{}_1, \cdots, x^{p-1}{}_n)$. Un calcul immédiat montre que la condition $\bar{Q}^k{}_f = 0$ pour $k > (m-n)/2$ est équivalente à $(\bar{p}^k{}_f)^h = 0$ modulo $p$ pour $k > (m-n)/2$ et qu'alors $\bar{Q}_f{}^{(m-n)/2} = (\bar{p}_f{}^{(m-n)/2})^h$ modulo $p$, où $h = (p-1)/2$.

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# FINITE GROUPS ADMITTING A FIXED-POINT-FREE AUTOMORPHISM OF ORDER 4.*

By Daniel Gorenstein and I. N. Herstein.

Recently, in a remarkable piece of work [4, 5] John Thompson has proved a result which implies as an immediate corollary the well-known Frobenius conjecture, namely that a finite group admitting a fixed-point-free automorphism (i. e., leaving only the identity element fixed) of prime order must be nilpotent. However, non-nilpotent groups are known which admit fixed-point-free automorphisms of composite order. In all these cases one notices that the groups in question are solvable. Although the sample is rather restricted, it is not too unnatural to ask whether the condition that a finite group admit such an automorphism is strong enough to force solvability of the group. This question is related to another problem, which seems equally difficult, which asks whether a finite group containing a cyclic subgroup which is its own normalizer must be composite.

In the present paper we shall prove that a group $G$ possessing a fixed-point-free automorphism of order 4 is solvable. Although many of the ideas used carry over to the case in which $\phi$ has order $pq$, and especially $2q$, our key lemmas use the fact that $\phi$ has order 4 in a crucial way.

The proof depends upon a theorem of Philip Hall which asserts that a finite group $G$ is solvable if for every factorization of $o(G)$ into relatively prime numbers $m$ and $n$, $G$ contains a subgroup of order $m$. We show (Lemma 7) that a group $G$ which has a fixed-point-free automorphism of order 4 satisfies the conditions of Hall's theorem.

Once we know that $G$ is solvable it is not difficult to prove that its commutator subgroup is nilpotent (Theorem 2). This fact was also observed by Thompson.

Graham Higman has shown [3] that there is a bound to the class of a $p$-group $P$ which possesses an automorphism $\phi$ of prime order $q$ without fixed-points. This does not carry over to automorphisms of composite order, for at the end of the paper we give an example due to Thompson of a family of $p$-groups of arbitrary high class each of which admits a fixed-point-free automorphism of order 4.

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**1.** We begin by recalling a few well-known elementary results concerning a finite group $G$ which admits a fixed-point-free automorphism $\phi$ of order $n$ and in particular when $n=4$. First of all, for any prime $p\,|\,o(G)$ there is a unique $p$-Sylow subgroup $P$ of $G$ which is invariant under $\phi$. We shall call $P$ the *canonical* $p$-Sylow subgroup of $G$ (with respect to $\phi$). Furthermore, for any $x$ in $G$ we have the relation $x\phi(x)\phi^2(x)\cdots\phi^{n-1}(x)=1$.

If $n=4$, each orbit under $\phi$ except for that consisting of the identity contains either 2 or 4 elements, hence $G$ is necessarily of odd order. The set of elements of $G$ left fixed by $\phi^2$ is a $\phi$-invariant subgroup of $G$, which we denote by $F$. If $F\neq 1$, the restriction of $\phi$ to $F$ is an automorphism of $F$ of order 2 without non-trivial fixed elements. This implies that $F$ is Abelian and that $\phi(f)=f^{-1}$ for all $f$ in $F$. Finally, we shall denote by $I$ the set of all $h$ in $G$ for which $\phi^2(h)=h^{-1}$. It is worth observing that $I$ need not be a subgroup of $G$.

Throughout the paper $G$ will denote a finite group having a fixed-point-free automorphism $\phi$ of order 4, $F$ will denote the subgroup left elementwise fixed by $\phi^2$ and $I$ the subset consisting of those elements of $G$ which are mapped into their inverses by $\phi^2$.

LEMMA 1. $G=FI=IF$.

*Proof.* If $z=x^{-1}\phi^2(x)$ for some $x$ in $G$, then $\phi^2(z)=\phi^2(x^{-1})\phi^4(x)$ $=\phi^2(x^{-1})x=z^{-1}$, whence $z\in I$. Furthermore, $x^{-1}\phi^2(x)=y^{-1}\phi^2(y)$ implies that $\phi^2(xy^{-1})=xy^{-1}$ and hence that $xy^{-1}\in F$. Thus $I$ contains at least $[G:F]$ elements.

To complete the proof, it will clearly suffice to show that distinct elements of $I$ lie in distinct right (or left) cosets of $F$. If $h_2=fh_1$, $h_1,h_2\in I$, $f\in F$, it follows by applying $\phi^2$ that $h_2^{-1}=fh_1^{-1}$. Combining this with the previous relation gives $h_1^{-1}fh_1=f^{-1}$. Since $G$ is of odd order, this forces $f=1$ and hence $h_1=h_2$. Similarly, we show that $h_2=h_1f$ implies $h_1=h_2$.

LEMMA 2. *If $f_1$, $f_2$ in $F$ are conjugate in $G$, then $f_1=f_2$.*

*Proof.* Suppose $xf_1x^{-1}=f_2$. Since $F$ is Abelian, we may assume without loss, in view of Lemma 1, that $x\in I$. Applying $\phi^2$ gives $x^{-1}f_1x=f_2$, whence $x^2$ centralizes $f_1$. Since $G$ is of odd order, $x$ centralizes $f_1$, and consequently $f_1=f_2$.

As an immediate corollary, we obtain

LEMMA 3. *Any subgroup of $F$ is in the center of its normalizer.*

LEMMA 4. *If $h\in I$, $h$ commutes with $\phi(h)$.*

*Proof.* This lemma follows at once from the relations $h\phi(h)\phi^2(h)\phi^3(h)=1$ and $\phi^2(h)=h^{-1}$.

LEMMA 5. *For any $p|o(G)$, $F$ normalizes the canonical $p$-Sylow subgroup $P$ of $G$.*

*Proof.* If $F \cap P=1$, $\phi^2$ is an automorphism of $P$ of order 2 without non-trivial fixed elements, whence $\phi^2(x)=x^{-1}$ for all $x \in P$. Thus $P \subset I$. Let $f \in F$ and consider $P'=fPf^{-1}$. If $y=fxf^{-1}$, with $x \in P$, $\phi^2(y)=fx^{-1}f^{-1}=y^{-1}$, which implies that $P' \subset I$.

Suppose $P' \neq P$; choose $y$ in $P'$ and not in $P$. The subgroup generated by $y$ and its image under $\phi$ is $\phi$-invariant and, since $y \in I$, it follows from the preceding lemma that this subgroup is a $p$-group. Let $P_1$ be a maximal $\phi$-invariant $p$-group containing $y$. If $P_1$ were not a $p$-Sylow subgroup of $G$, the unique $\phi$-invariant $p$-Sylow subgroup $P_2$ of $N_G(P_1)$ would have order greater than $o(P_1)$ and would contain $P_1$ and consequently $y$. Since this would contradict our choice of $P_1$, $P_1$ must be a $p$-Sylow subgroup of $G$. Since $P$ is the only $\phi$-invariant $p$-Sylow subgroup of $G$, $P_1=P$, which is impossible since $y \in P_1$, $y \notin P$. We conclude that $P'=fPf^{-1}=P$. Since $f$ was arbitrary, $F \subset N_G(P)$.

Suppose, on the other hand, that $F \cap P \neq 1$. In this case we shall prove the lemma by induction on $o(G)$. Since $F$ is Abelian, $F \cap P$ is a $\phi$-invariant $p$-group which is normalized by $F$. If $P_1$ is a maximal $\phi$-invariant $p$-group which is normalized by $F$, it follows first of all as in the preceding paragraph that $P_1 \subset P$. Suppose $P_1 < P$. We must have $N_G(P_1)=G$, for otherwise by induction $F$ normalizes the unique $\phi$-invariant $p$-Sylow subgroup $P_2$ of $N_G(P_1)$ and $o(P_2) > o(P_1)$. Thus $P_1 \lhd G$. Set $\bar{G}=G/P_1$ and let $\bar{\phi}$ be the image of $\phi$ on $\bar{G}$. $\bar{\phi}$ has no non-trivial fixed elements and is of order 2 or 4. If $\bar{P}$, $\bar{F}$ denote the images of $P$, $F$ in $\bar{G}$, it follows by induction (or from the fact that $\bar{G}$ is Abelian in the case $\bar{\phi}^2=1$) that $\bar{F} \subset N_{\bar{G}}(\bar{P})$. Thus $F \subset N_G(P)$.

LEMMA 6. *If $A$, $B$ are two $\phi$-invariant subgroups of $G$ which are each normalized by $F$, then $ABF$ is a $\phi$-invariant subgroup of $G$ of order dividing $o(A)o(B)o(F)$.*

*Proof.* Since $BF$ is a subgroup, $ABF$ will be a subgroup if $(BF)A=A(BF)$. Since $F$ normalizes $A$, it will suffice to show that $BA \subset ABF$.

Since $A$ is $\phi$-invariant, it follows from Lemma 1 applied to $A$ that for any $a$ in $A$, $a=a'f_1$, where $a' \in I \cap A$ and $f_1 \in F \cap A$. Similarly, for any $b$

in $B$, $b = f_2 b'$, $f_2 \in F \cap B$ and $b' \in I \cap B$. Clearly, $ba = f_2 b' a' f_1 \in ABF$ if and only if $b'a' \in ABF$.

Now $b'^{-1}a'^{-1} = fh$ for some $f$ in $F$, $h$ in $I$; applying $\phi^2$ gives $b'a' = fh^{-1}$. Since $h^{-1} = a'b'f$ from the first relation, $b'a' = fa'b'f = a''b''f^2$, where $a'' \in A$, $b'' \in B$. Thus $ABF$ is a subgroup as asserted. The remaining parts of the lemma are immediate.

LEMMA 7. *Let $p_1, p_2, \cdots, p_k$ be a set of primes dividing $o(G)$ and let $P_1, P_2, \cdots, P_k$ be the corresponding canonical Sylow subgroups of $G$. Then $P_1 P_2 \cdots P_k$ is a subgroup of $G$.*

*Proof.* By induction on $k$ we may assume that $H = P_1 P_2 \cdots P_{k-1}$ is a subgroup of $G$. Clearly $H$ is $\phi$-invariant. By Lemma 5 $F \subset N_G(H)$ and $F \subset N_G(P_k)$, so that $S = HP_kF$ is a $\phi$-invariant subgroup of $G$ by Lemma 6. Since $o(S) \mid o(H)o(P_k)o(F)$, a $q$-Sylow subgroup $Q$ of $F$ for any prime $q \neq p_i$, $i = 1, 2, \cdots, k$, is a $q$-Sylow subgroup of $S$. By Lemma 3 $Q$ is in the center of its normalizer in $S$, so that by a well-known theorem of Burnside $S$ possesses a normal $q$-complement $L_q$. Since $L_q$ consists of the elements of $S$ of order prime to $q$, $L_q$ contains $H$ and $P_k$. Repeating this argument for each such prime $q \mid o(F)$, we readily conclude that $\bigcap_{\substack{q \mid o(F) \\ q \neq p_i}} L_q = HP_k = P_1 P_2 \cdots P_k$, which, being an intersection of subgroups, is a subgroup.

Lemma 7 leads at once to our main result.

THEOREM 1. *If $G$ is a finite group admitting an automorphism of order 4 leaving only the identity element of $G$ fixed, then $G$ is solvable.*

*Proof.* It follows from Lemma 7 that for any factorization of $o(G)$ into the product of relatively prime numbers $m$ and $n$, $G$ contains a subgroup of order $m$. By a theorem of Philip Hall ([2], Theorem 9.3.3, p. 144), this implies that $G$ is solvable.

**2.** We shall now examine the structure of $G$ more closely, For our main result we need several lemmas.

LEMMA 8. *If $G = HM$, where $H$ is nilpotent, normal in $G$, $(o(H), o(M)) = 1$, $M$ is invariant under $\phi$ and $M \cap F = 1$, then $G = H \times M$.*

*Proof.* Let $\Phi(H)$ be the Frattini subgroup of $H$ and set $\bar{G} = G/\Phi(H) = \bar{H}\bar{M}$. Since $(o(H), o(M)) = 1$, it follows from the properties of the Frattini subgroup that $\bar{G} = \bar{H} \times \bar{M}$ implies $G = H \times M$. Hence, without loss, we may assume that $H$ is elementary Abelian. Since $\phi^2$ leaves only the

identity element of $M$ fixed, $M$ is Abelian. If either $H$ contains two disjoint $\phi$-invariant subgroups normal in $G$ or $\phi$ does not act irreducibly on $M$, the lemma follows easily by induction. Hence we may assume that $\phi$ acts irreducibly on $M$ and no proper $\phi$-invariant subgroup of $H$ is normal in $G$. In particular, this implies that $H$ is an elementary Abelian $p$-group for some prime $p$.

The holomorph of $\phi$ and $M$ is represented irreducibly on $H$ regarded as a vector space over the prime field $K_p$ with $p$ elements. Let $H^*$ be the corresponding vector space over the algebraic closure $K^*_p$ of $K_p$. If $M$ does not centralize $H$, it follows from Lemma 3. 1 of [1] that with respect to a suitable basis of $H^*$ the matrix of $\phi$ assumes the form

$$\begin{bmatrix} \phi_1 & & & 0 \\ & \phi_2 & & \\ & & \ddots & \\ 0 & & & \phi_s \end{bmatrix},$$

where

$$\phi_i = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ b_i & 0 & 0 & 0 \end{bmatrix}$$

with $b_i \in K^*_p$. Since $\phi^4 = 1$, $\phi_i^4 = 1$ and hence $b_i = 1$ for all $i$. But this means that 1 is a characteristic root of $\phi$ and hence that $\phi$ leaves some element of $H$ other than the identity fixed. This contradiction forces $M$ to centralize $H$, and consequently $G = H \times M$, as asserted.

LEMMA 9. *If $G = HM$, where $H$ is nilpotent, normal in $G$, $(o(H), o(M)) = 1$, $M$ is invariant under $\phi$ and $C_G(H) \subset H$, then $M \subset F$.*

*Proof.* By Theorem 1 $G$ and hence $M$ is solvable. Let $K$ be a maximal $\phi$-invariant normal subgroup of $M$. By induction applied to $HK$, $K \subset F$. Let $\bar{M} = M/K$ and let $\bar{\phi}$ be the image of $\phi$ on $\bar{M}$. If $\bar{\phi}^2 = 1$ on $\bar{M}$, it follows readily that $M \subset F$. Hence we may assume that $\bar{\phi}$ has order 4 on $\bar{M}$. Since $\bar{M}$ is elementary Abelian and $\bar{\phi}$ acts irreducibly on $\bar{M}$, $\bar{\phi}^2(\bar{y}) = \bar{y}^{-1}$ for all $\bar{y}$ in $\bar{M}$. Thus for all $y$ in $M$, $\phi^2(y) = y^{-1}z$, $z \in K$. Now if $x \in K$, $yxy^{-1} = x'$ for some $x' \in K$. Applying $\phi^2$ gives $y^{-1}zxz^{-1}y = x'$. Since $K$ is abelian we easily conclude that $y^2$ and consequently $y$ centralizes $x$. Hence $K$ is in the center of $M$.

As in the previous lemma we may assume without loss of generality that

$H$ is elementary Abelian. By Lemma 1 applied to $M$, $M = (F \cap M)(I \cap M)$, and under our present assumptions $I \cap M \neq 1$. By induction we may suppose that no $\phi$-invariant proper subgroup of $H$ is normal in $H$; and we shall then derive a contradiction by showing that $I \cap M$ centralizes $H$.

Now $C_G(K)$ is $\phi$-invariant and contains $M$, since $K$ is in the center of $M$. Since $H_1 = H \cap C_G(K) \lhd C_G(K), N_G(H_1)$ contains $M$ and $H$, whence $H_1 \lhd G$. Since $H_1$ is invariant under $\phi$, the minimality of $H$ implies that either $H_1 = 1$ or $H_1 = H$.

Suppose first that $H_1 = 1$. Since $K \subset F$ and $F$ is Abelian $H \cap F \subset C_G(K)$, whence $H \cap F = 1$ and consequently $H \subset I$. If $y \in M \cap I$, $x \in H$, $yxy^{-1} = x'$ for some $x'$ in $H$. Applying $\phi^2$, we obtain $y^{-1}x^{-1}y = x'^{-1}$, which together with the preceding relation implies that $x$ and $y$ commute. Thus $I \cap M \subset C_G(H)$ as asserted.

On the other hand, if $H_1 = H$, $K$ centralizes $H$, whence $K = 1$ and $I \cap M = M$. Since $M \cap F = 1$, $G = H \times M$ by Lemma 8. Thus $I \cap M$ centralizes $H$, completing the proof.

LEMMA 10. *$G$ has $p$-length 1 for all $p \mid o(G)$.*

*Proof.* Since $G$ is solvable by Theorem 1, the statement of the lemma is meaningful. The proof will be by induction on $o(G)$. Let $M$ be the maximal normal subgroup of $G$ of order prime to $p$, and assume first that $M \neq 1$. $M$ is $\phi$-invariant since it is characteristic in $G$. Let $\bar{\phi}$ be the image of $\phi$ on $\bar{G} = G/M$. If $\bar{\phi}^2 = 1$ on $\bar{G}$, $\bar{G}$ is Abelian. If $\bar{\phi}$ has order 4 on $\bar{G}$, it follows by induction that $\bar{G}$ has $p$-length 1 and hence that a $p$-Sylow subgroup $\bar{P}$ of $\bar{G}$ is normal in $\bar{G}$. In either case we conclude that $G$ has $p$-length 1. We may therefore suppose that $M = 1$.

Let $P_1$ be the maximal normal $p$-group of $G$ and consequently $\phi$-invariant. Let $P$ be the canonical $p$-Sylow subgroup of $G$ and $\tilde{P}$ its image in $\tilde{G} = G/P_1$. If $\tilde{K}$ is the maximal normal subgroup of $\tilde{G}$ of order prime to $p$, it follows by induction that the image of $\tilde{P}$ in $\tilde{G}/\tilde{K}$ is normal in $\tilde{G}/\tilde{K}$ whence $\tilde{P}\tilde{K} \lhd \tilde{G}$. If $\tilde{P}\tilde{K} < \tilde{G}$, its inverse image $G_0 < G$, and hence by induction has $p$-length 1.

Since $P_1$ contains its own centralizer in $G$ ([2], Theorem 18.4.4, p. 332), $G_0$ contains no non-trivial normal subgroups of order prime to $p$ and hence $P \lhd G_0$. But then $\tilde{P} \lhd \tilde{P}\tilde{K}$, and since $\tilde{P}$ is characteristic in $\tilde{P}\tilde{K}$, we conclude that $\tilde{P} \lhd \tilde{G}$, whence $P \lhd G$.

We may therefore assume that $\tilde{G} = \tilde{P}\tilde{K}$. The inverse image $G_1$ of $\tilde{K}$ is of the form $P_1K$, where $K$ has order prime to $p$. Since $G_1$ is solvable, any two subgroups of $G_1$ of order $o(K)$ are conjugate ([2], Theorem 9.3.1, p. 141). One can now show by the same argument which proves the existence of

canonical $p$-Sylow subgroups that there exists a unique conjugate of $K$ in $G$, which is invariant under $\phi$. Without loss we may assume $K$ itself is $\phi$-invariant. Since $C_{G_1}(P_1) \subset P_1$, the previous lemma implies that $K \subset F$. But then by the argument of the first paragraph of the lemma $\bar{K}$ is in the center of $\bar{G}$, whence $\bar{P} \lhd \bar{G}$ and $P \lhd G$.

THEOREM 2. *If $G$ possesses an automorphism $\phi$ of order 4 leaving only the identity element fixed, then the commutator subgroup of $G$ is nilpotent.*

*Proof.* $G$ is solvable by Theorem 1. Assume first that $G$ contains two minimal $\phi$-invariant normal subgroups $N_1$ and $N_2$. Since the image of $\phi$ on $G/N_1$ and $G/N_2$ has no non-trivial fixed elements, the commutator subgroups $[G/N_i, G/N_i]$ of $G/N_i$, $i = 1, 2$, are nilpotent by induction. Let $H_i$ be the inverse image of $[G/N_i, G/N_i]$ in $G$ and set $H = H_1 \cap H_2$. Clearly, $H \lhd G$ and $[G, G] \subset H$. Furthermore, if $x$ and $y$ are elements of relatively prime order in $H$, their images in $G/N_i$, $i = 1, 2$, commute, and hence $y^{-1}xyx^{-1} \in N_1 \cap N_2$. Since $N_1$ and $N_2$ are distinct minimal normal $\phi$-invariant subgroups of $G$, $N_1 \cap N_2 = 1$, and consequently $x$, $y$ commute. Thus $H$ and hence $[G, G]$ is nilpotent.

We may therefore suppose that $G$ contains a unique minimal normal $\phi$-invariant subgroup $N_1$. $N_1$ is a $p$-group for some prime $p$ and $G$ contains no non-trivial normal subgroups of order prime to $p$. Since $G$ has $p$-length 1 by Lemma 10, a $p$-Sylow subgroup $P$ of $G$ is normal in $G$. Now $C_G(P) \lhd G$ and $C_G(P) = Z(P) \times K$, where $K$ has order prime to $p$. Since $K$ is characteristic in $C_G(P)$, $K$ is normal in $G$, whence $K = 1$ and $C_G(P) \subset P$. Furthermore, $G = PM$ for some subgroup $M$ of $G$, and we may assume $M$ is invariant under $\phi$. Since $(o(P), o(M)) = 1$, we can apply Lemma 9 to conclude that $M \subset F$. Thus $M$ is Abelian, and consequently $[G, G] \subset P$ is nilpotent.

**3.** We conclude with an example of a family of $p$-groups of arbitrarily high class each of which has a fixed-point free automorphism of order 4. Let $p$ be any prime such that $p \equiv 1 \pmod 4$ and let $P_1$ be an elementary Abelian $p$-group of order $p^t$, where $t = p^d$, $d$ an arbitrary integer, and let $x_1, x_2, \cdots, x_t$ be a basis for $P_1$. We construct an extension of $P_1$ by adjoining a new letter $y$ satisfying the relations:

$$(*) \qquad y^t = 1, \; yx_iy^{-1} = x_ix_{i+1}, \; i = 1, 2, \cdots, t-1, \; yx_ty^{-1} = x_t.$$

These relations define a $p$-group of order $tp^t$ and of class $t$.

Since $p \equiv 1 \pmod 4$, there is an integer $\alpha$ such that $\alpha^2 \equiv -1 \pmod p$.

We define an automorphism $\theta$ of $P$ by setting $\theta(y) = y^{-1}$, $\theta(x_t) = x_t^{\alpha}$. For $\theta$ to be an automorphism, its value on $x_i$ must be such that

$$(**) \qquad y^{-1}\theta(x_i)y = \theta(x_i)\theta(x_{i+1}).$$

Assume $\theta$ has been defined on $x_j$ for $j > i$ and that $\theta(x_j)$ is in the subgroup generated by $x_j, x_{j+1}, \cdots, x_t$. We shall show that $\theta(x_i)$ can be defined satisfying (**) and subject to the restriction $\theta(x_i) = x_i^{a_i}x_{i+1}^{a_{i+1}} \cdots x_t^{a_t}$. It follows at once from (**) that we have $x_{i+1}^{-a_i}x_{i+2}^{-a_{i+1}} \cdots x_t^{-a_{t-1}} = y\theta(x_{i+1})y^{-1}$. Since these relations have a solution for $a_i, a_{i+1}, \cdots, a_{t-1}$ (for any choice of $a_t$), the automorphism $\theta$ exists.

Regarding $P_1$ as a vector space, it is easy to see that the matrix of $\theta$ with respect to the basis $x_1, x_2, \cdots, x_t$ has the form $\alpha D + N$, where $D = \operatorname{diag}(1, -1, 1, -1, \cdots, 1)$ and $N$ is a strictly triangular matrix. It follows that the order of $\theta$ on $P_1$ is $4p^s$ for some $s \leqq d$. Setting $\phi = \theta^{p^s}$, $\phi$ has order 4 on $P_1$ and since $\phi(y) = y^{-1}$, $\phi$ has order 4 on $P$. The characteristic roots of $\phi$ are $\pm\alpha$, the same as those of $\theta$. Since $\alpha \neq \pm 1$, $\phi$ leaves only the identity element of $P_1$ fixed. Since $\phi(y) = y^{-1}$, this implies that $\phi$ leaves only the indentity element of $P$ fixed.

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# ON INDUCED REPRESENTATIONS.*

By Robert J. Blattner.

**1. Introduction.** The purpose of this paper is threefold: to lay the foundations of a theory of induced representations of (not necessarily separable) locally compact groups, to prove a sharpened form of an intertwining number theorem due to Bruhat, and to prove a disjointness theorem for representations of Lie groups induced from compact subgroups. G. W. Mackey in [10] developed a notion of induced representation when the group $G$ induced up to is separable and the inducing representation $L$ is in a separable Hilbert space $\mathfrak{V}$. Because the construction rested on the choice of a quasi-invariant measure in the relevant homogeneous space and because the existence of such measures is problematical when $G$ is not separable, this definition is not suitable for generalization to the non-separable case. The definition we employ, which is equivalent to Mackey's when $G$ and $\mathfrak{V}$ are separable, is a modification of the one used by F. Bruhat in [3]. The chief novelty is the way in which the Hilbert space structure is defined on the function space $\mathfrak{H}$ in which the induced representation $U^L$ operates, a way which owes much to Mackey's notion of intrinsic Hilbert space. Section 2 deals with the basic properties of induced representations, concluding with the theorem on induction in stages (Theorem 1).

In Sections 3 through 5 we are concerned with the problem, already considered by Bruhat, of finding the intertwining number of two inductions when $G$ is a Lie group. Our method is based on two facts: (1) any function $f$ in $\mathfrak{H}$ which is in the domain $\mathfrak{H}_\infty$ of all the operators of the differential representation $\partial U^L$ of the enveloping algebra $\mathfrak{E}$ of the Lie algebra of $G$ is essentially continuous; (2) if $X \in \mathfrak{E}$ is elliptic as a left invariant differential operator on $G$ and has sufficiently high order, then $f(e)$ may be estimated in terms of $\partial U^L(X)f$ when $f \in \mathfrak{H}_\infty$. Our estimate of the intertwining number (Theorem 3) is in terms of the dimension of a certain space of distributions, the order of which is usually much lower than that needed in Bruhat's theorem. This would seem to result from our use of *unitary* representations throughout our discussion and our replacement of Schwartz's kernel theorem by the facts on elliptic operators mentioned above. The importance in other connections

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of the theory of elliptic operators for Lie group representation theory has been established in recent papers by Stinespring ([15]) and Nelson and Stinespring ([12]). In Section 6 we prove a disjointness criterion for inductions to Lie groups of finite dimensional representations of compact subgroups (Theorem 4). This is accomplished by applying direct integral techniques to the results of Section 5. The paper concludes with an example in Section 7.

*Notation.* If $f$ is a numerical function on a set $S$, $\| f \|_S = \mathrm{LUB}[| f(s)| : s \in S]$. If $O$ is an open subset of a topological space and if $\mathfrak{X}$ is a topological linear space, $C(O;\mathfrak{X})$ denotes the class of all continuous functions from $O$ to $\mathfrak{X}$ and $C_0(O;\mathfrak{X})$ denotes the subclass of $C(O;\mathfrak{X})$ consisting of functions with compact support. A superscript $\infty$ (resp. $+$) indicates restriction to indefinitely differentiable (resp. non-negative) functions. If $\mathfrak{Y}$ is another topological linear space, $\mathfrak{L}(\mathfrak{X};\mathfrak{Y})$ is the space of all continuous linear maps of $\mathfrak{X}$ into $\mathfrak{Y}$ equipped with the topology of bounded convergence.

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**2. Induced representations.** Let $G$ be a locally compact group, $\Gamma$ a closed subgroup of $G$, and $L$ a unitary representation of $\Gamma$ on the Hilbert space $\mathfrak{V}$. Let right Haar measure be chosen in $G$ and $\Gamma$, and let their respective modular functions be $\Delta$ and $\delta$. Let $\mathfrak{F}^*$ be the set of all functions $f$ from $G$ to $\mathfrak{V}$ such that: (1) $f(\cdot)$ is Bourbaki measurable (see [2], p. 180); (2) $f(\xi x) = \Delta(\xi)^{-\frac{1}{2}}\delta(\xi)^{\frac{1}{2}}L_\xi f(x)$ whenever $\xi \in \Gamma$ and $x \in G$; (3) $\| f(\cdot)\|^2$ is locally integrable. Let $M = G/\Gamma$ (right coset space) and let $\pi$ be the canonical projection of $G$ on $M$. As in Lemma 1.5 of [10], $\| f(\cdot)\|^2$ defines a positive Radon measure $\mu_f$ on $M$ via the equation

$$\int_G \| f(x)\|^2 g(x)\,dx = \int_M (\tau g)(p)\,d\mu_f(p),$$

where $g \in C_0(G)$ and $(\tau g)(\pi(x)) = \int_\Gamma g(\xi x)\,d\xi$. Set $\| f \| = \mu_f(M)^{\frac{1}{2}}$ and $\mathfrak{F} = [f \in \mathfrak{F}^* : \| f \| < \infty]$. One easily sees that $\mathfrak{F}^*$and $\mathfrak{F}$ are linear spaces.

If $f, g \in \mathfrak{F}$, then $(f(\cdot), g(\cdot))$ is Bourbaki measurable ([2], Proposition 10, p. 193) and $(f(\cdot), g(\cdot))$ defines a finite complex valued Radon measure on $M$, call it $\mu_{f,g}$, in the same way that $\| f(\cdot)\|^2$ defines $\mu_f$. We set $(f,g) = \mu_{f,g}(M)$. $(\cdot,\cdot)$ is a positive semi-definite Hermitian form on $F$ and $\| f \| = (f,f)^{\frac{1}{2}}$. Clearly $\| f \| = 0$ if and only if $f(\cdot) = 0$ locally almost everywhere (l. a. e.). Thus, if we set $\mathfrak{H} = \mathfrak{F}/[f \in \mathfrak{F} : f(\cdot) = 0 \text{ l. a. e.}]$, we may

transport $(\cdot,\cdot)$ to $\mathfrak{H}$ and $\mathfrak{H}$ is then a pre-Hilbert space. (In the sequel, we shall often willfully confuse $\mathfrak{F}$ and $\mathfrak{H}$.)

In order to show completeness, we need the following estimate, which is also useful in other connections.

LEMMA 1. *For each compact subset $K$ of $G$, there is a constant $\Lambda_K$ such that, for all $f \in \mathfrak{F}$,*

$$\int_K \| f(x) \| \, dx \leqq \Lambda_K \| f \|.$$

*Proof.* Choose $g \in C_0^+(G)$ such that $g = 1$ on $K$. Then

$$\int_K \| f(x) \|^2 \, dx \leqq \int_G g(x) \| f(x) \|^2 \, dx = \int_M (\tau g)(p) \, d\mu_f(p) \leqq \| f \|^2 \, \| \tau g \|_M.$$

Set $\Lambda_K = (\| \tau g \|_M \int_K dx)^{\frac{1}{2}}$ and apply the Schwarz inequality.

PROPOSITION 1. *$\mathfrak{H}$ is complete.*

*Proof* (after Riesz-Fischer). Let $\{f_n\}$ be a Cauchy sequence in $\mathfrak{F}$. As usual it suffices to show $\{f_n\}$ has a limit in $\mathfrak{F}$ under the added assumption that $\| f_n - f_{n+1} \| < 2^{-n}$. First we show that for locally almost all $x \in G$, $\operatorname{Lim}_{n\to\infty} f_n(x)$ exist in $\mathfrak{V}$. In fact, let $K$ be a compact subset of $G$. Lemma 1 tells us that $\int_K \| f_n(x) - f_{n+1}(x) \| \, dx < 2^{-n}\Lambda_K$, whence

$$\int_K \left(\sum_1^\infty \| f_n(x) - f_{n+1}(x) \|\right) dx < \Lambda_K.$$

Therefore for locally almost all $x \in K$, $\{f_n(x)\}$ is Cauchy in $\mathfrak{V}$. Set $f(x) = \operatorname{Lim}_{n\to\infty} f_n(x)$ or 0 according as this limit exists or not.

Clearly $f$ satisfies properties (1) and (2) for $\mathfrak{F}^*$. We must show that $f$ satisfies (3), that $\| f \| < \infty$, and that $\| f_n - f \| \to 0$. Let $g \in C_0^+(G)$.

Iterating the parallelogram identity for $\mathfrak{V}$, we see that

$$\int \| f_r(x) - f_{r+k}(x) \|^2 g(x) \, dx \leqq \sum_{j=1}^\infty 2^j \int \| f_{r+j-1}(x) - f_{r+j}(x) \|^2 g(x) \, dx$$

$$\leqq \sum_{j=1}^\infty 2^j \| f_{r+j-1} - f_{r+j} \|^2 \, \| \tau g \|_M < 2^{-2r+2} \| \tau g \|_M.$$

By Fatou's lemma, $\int \| f_r(x) - f(x) \|^2 g(x) \, dx \leqq 2^{-2r+2} \| \tau g \|_M$. Letting $K$ be compact in $G$ and setting $g = 1$ on $K$, we see that $\| f_r(\cdot) - f(\cdot) \|^2$ is

integrable on $K$. Hence $f \in \mathfrak{F}^*$. Letting $g$ be arbitrary in $C_0^+(G)$, we see that $\| f_r - f \| \leqq 2^{-2r+2}$, whence $f \in \mathfrak{F}$. Finally, $\| f_r - f \| \to 0$.

The following is a restricted form of Bruhat's version ([3], § 4) of a map first introduced by Mackey ([10], § 3). Let $f \in C_0(G)$ and $v \in \mathfrak{V}$. Form

$$\epsilon(f,v)(x) = \int_\Gamma \delta(\xi)^{-\frac{1}{2}}\Delta(\xi)^{\frac{1}{2}} f(\xi x) L_\xi^{-1} v \, d\xi.$$

The support of $\epsilon(f,v)$ is contained in $\Gamma K$ if the support of $f$ is $K$. Letting $\mathfrak{F}_0$ denote the subspace of $\mathfrak{F}^*$ consisting of functions which are continuous and have compact support modulo $\Gamma$, we have $\mathfrak{F}_0 \subseteq \mathfrak{F}$. Clearly $\epsilon(f,v) \in \mathfrak{F}_0$ and is bilinear. Two important facts about $\epsilon$ are summarized in the following lemma (cf. [10], Lemma 3.1 and 3.5).

LEMMA 2. (a) *If $K$ is the support of $f$, $\| \epsilon(f,v) \| \leqq \Lambda_K \| f \|_G \| v \|$.* (b) *If $\mathfrak{D}$ is total in $\mathfrak{V}$, then $\epsilon(C_0(G) \times \mathfrak{D})$ is total in $\mathfrak{H}$.*

*Proof.* (a) Choose $h \in C_0(G)$ so that $\tau h = 1$ on $\pi(K)$. Let $g \in \mathfrak{F}$. Then

$$(\epsilon(f,v),g) = \int_G h(x)(\epsilon(f,v)(x), g(x))dx$$
$$= \int_G \int_\Gamma h(x)\delta(\xi)^{-\frac{1}{2}}\Delta(\xi)^{\frac{1}{2}} f(\xi_x)(L_\xi^{-1} v, g(x))d\xi dx.$$

Using the Fubini theorem and the invariance properties of Haar measure, this becomes $\int_G \int_\Gamma h(\xi x) f(x)(v, g(x)) d\xi dx = \int_G f(x)(v,g(x))dx$ by the choice of $h$. Lemma 1 and the Schwarz inequality give $|(\epsilon(f,v),g)| \leqq \Lambda_K \| f \|_G \| g \| \| v \|$ and our result follows.

(b) Suppose $(\epsilon(f,v),g) = 0$ for all $f \in C_0(G)$, $v \in \mathfrak{D}$.

The calculation in (a) shows that for each $v \in \mathfrak{D}$, $(v, g(\cdot)) = 0$ l. a. e. Using [2], Proposition 10, p. 193, we see that $g(\cdot) = 0$ l. a. e.

*Remark.* If $G$ is a Lie group, the above proof shows that $\epsilon(C_0^\infty(G) \times \mathfrak{V})$ is total in $\mathfrak{H}$.

For any function $f$ on $G$, define $R_y f$ by $(R_y f)(x) = f(xy)$. Clearly $R_y$ carries $\mathfrak{F}^*$ into $\mathfrak{F}^*$. We assert that $\| R_y f \| = \| f \|$ for $y \in G$, $f \in \mathfrak{F}^*$. In fact, let $g \in C_0(G)$. Then

$$\int_M (\tau g)(p) d\mu_{R_y f}(p) = \int_G \| (R_y f)(x) \|^2 g(x) dx$$
$$= \int_G \| f(x) \|^2 (R_{y^{-1}} g)(x) dx = \int_M (\tau R_{y^{-1}} g)(p) d\mu_f(p)$$
$$= \int_M (R'_{y^{-1}} \tau g)(p) d\mu_f(p),$$

where $(R'_y h)(p) = h(py)$ for any function $h$ on $M$ and any $y \in G$. Taking the sup over all g such that $0 \leqq \tau g \leqq 1$, we have our assertion.

For $y \in G$, let $U^L{}_y$ be the unitary map of $\mathfrak{H}$ onto itself defined by $R_y$.

PROPOSITION 2. *The map $U^L$ sending $y \to U^L{}_y$ is a continuous unitary representation of $G$ in $\mathfrak{H}$.*

*Proof.* The representation property is clear. For continuity, let $f, g \in \mathfrak{F}_0$ and let $N$ be a compact neighborhood of $e$ in $G$. The supports of all the measures $\mu_{U_y f, g}$, $y \in N$, are contained in some common compact subset $K$ of $M$. Choose $h \in C_0(G)$ so that $\tau h = 1$ on $K$. Then, for $y \in N$,

$$(U_y f, g) = \int_G h(x)(f(xy), g(x))\,dx,$$

which is a continuous function of $y$ by standard theorems on integration. Our result follows because $\mathfrak{F}_0$ is dense in $\mathfrak{F}$ by Lemma 2.

$U^L$ is called the *representation of $G$ induced by $L$.* If it is not clear from the context what group is being up to, we shall write ${}_G U^L$.

*Remark.* In case $G$ and $M$ are separable, our definition of $U^L$ and Mackey's ([10]) are equivalent. In fact, let $\nu$ be a quasi-invariant measure on $M$ defined by a $\rho$-function $\rho$ ([10]), Lemma 1.4). If $f \in \mathfrak{F}_0$, it is readily seen that $\rho^{-\frac{1}{2}} f \in {}^{\mu}\mathfrak{H}^L$ ([10]), § 2) and that the map $f \to \rho^{-\frac{1}{2}} f$ is isometric. The image of $\mathfrak{F}_0$ under this map is easily seen to satisfy the conditions of Lemma 3.3 in [10]. Because of that lemma and our Lemma 2, the map extends to a unitary map of $\mathfrak{H}$ onto ${}^{\mu}\mathfrak{H}^L$, which sets up the required equivalence. We note that our definition of induced representation is fixed once Haar measure has been fixed in $G$ and $\Gamma$. Moreover, for different choices of Haar measure we obtain the same $\mathfrak{H}$ with the norm changed by a multiplicative constant. Thus the problem in [10] of showing that all ways of defining the induced representation are equivalent is trivialized.

We end this section by proving the theorem on induction in stages:

THEOREM 1. *Let $\Gamma_1$ and $\Gamma_2$ be closed subgroups of $G$ with $\Gamma_1 \subseteq \Gamma_2$. Let $L$ be a unitary representation of $\Gamma_1$ on $\mathfrak{V}$ and denote the inductions of $L$ to $\Gamma_1$ and $G$ by $M$ and $U$ respectively. Then $U$ is unitarily equivalent to $U^M$.*

*Proof.* Let $\delta_1$, $\delta_2$, and $\Delta$ be the modular functions for $\Gamma_1$, $\Gamma_2$, and $G$ respectively. Let $\mathfrak{F}^{(1)}$, $\mathfrak{F}^{(2)}$, and $\mathfrak{F}$ denote the spaces for the inductions from $\Gamma_1$ to $\Gamma_2$, $\Gamma_2$ to $G$, and $\Gamma_1$ to $G$ respectively, corresponding to the space $\mathfrak{F}$ in our construction. Let $f \in \mathfrak{F}_0$ with support in $\Gamma_1 K$, compact. For $\eta \in \Gamma_2$, $x \in G$, set $\hat{f}(\eta, x) = \delta_2(\eta)^{-\frac{1}{2}} \Delta(\eta)^{\frac{1}{2}} f(\eta x)$. Let $x$ be fixed. Then

$$\hat{f}(\xi\eta, x) = \delta_1(\xi)^{\frac{1}{2}} \delta_2(\xi)^{-\frac{1}{2}} L_\xi \hat{f}(\eta, x), \quad \xi \in \Gamma_1, \eta \in \Gamma_2.$$

Moreover $\hat{f}(\cdot, x)$ is continuous with support in $\Gamma_1(Kx^{-1} \cap \Gamma_2)$. Thus $\hat{f}(\cdot, x)$ is a member of $\mathcal{F}^{(1)}{}_0$, which we denote by $\hat{f}(x)$. Now

$$\hat{f}(\eta, \zeta x) = \delta_2(\zeta)^{\frac{1}{2}}\Delta(\zeta)^{-\frac{1}{2}}\hat{f}(\eta\zeta, x)$$

for $\eta, \zeta \in \Gamma_2$, $x \in G$, so that $\hat{f}(\zeta x) = \delta_2(\zeta)^{\frac{1}{2}}\Delta(\zeta)^{-\frac{1}{2}}M_\zeta f(x)$. The support of $\hat{f}(\cdot)$ is in $\Gamma_2 K$. To prove continuity, let $N$ be a compact neighborhood of $e$ in $G$ and choose $h \in C_0(G)$ so that $\int_{\Gamma_1} h(\xi x)\, d\xi = 1$ on $\Gamma_1 KN$. Then $\int_{\Gamma_1} h(\xi\eta x)\, d\xi = 1$ for $\eta \in \Gamma_1(KNx^{-1} \cap \Gamma_2)$. Hence

$$\| \hat{f}(x) - \hat{f}(y) \|^2 = \int_{\Gamma_2} h(\eta x) \| \hat{f}(\eta, x) - \hat{f}(\eta, y) \|^2\, d\eta$$
$$= \int_{\Gamma_2} \delta_2(\eta)^{-1}\Delta(\eta) h(\eta x) \| f(\eta x) - f(\eta y) \|^2\, d\eta$$

whenever $y^{-1}x \in N$, and the continuity of $\hat{f}$ follows from the uniform continuity of $f$ on compact sets. Thus $\hat{f}(\cdot)$ is a member of $\mathcal{F}^{(2)}{}_0$, which we denote by $\hat{f}$. Now $\| \hat{f}(x) \|^2 = \int_{\Gamma_2} \delta_2(\eta)^{-1}\Delta(\eta) h(\eta x) \| f(\eta x) \|^2\, d\eta$. Choose $k \in C_0(G)$ so that $\int_{\Gamma_2} k(\eta x)\, dy = 1$ on $\Gamma_2 K$. Then, using the Fubini theorem and the invariance of Haar measure,

$$\| \hat{f} \|^2 = \int_G k(x) \| \hat{f}(x) \|^2\, dx$$
$$= \int_G \int_{\Gamma_2} k(x)\delta_2(\eta)^{-1}\Delta(\eta) h(\eta x) \| f(\eta x) \|^2\, d\eta dx$$
$$= \int_{\Gamma_2} \int_G k(\eta^{-1}x)\delta_2(\eta)^{-1} h(x) \| f(x) \|^2\, dx d\eta$$
$$= \int_G h(x) \| f(x) \|^2 [\int_{\Gamma_2} k(\eta x)\, d\eta]\, dx = \| f \|^2$$

by the choice of $h$ and $k$. Thus $f \to \hat{f}$ is an isometry of $\mathcal{F}_0$ into $\mathcal{F}^{(2)}{}_0$.

We next assert that the image of $\mathcal{F}_0$ in $\mathcal{F}^{(2)}{}_0$ is dense. Let $g \in C_0(\Gamma_2)$, $h \in C_0(G)$, $v \in \mathcal{V}$, and set $k(x) = \int_{\Gamma_2} \delta_2(\zeta)^{-\frac{1}{2}}\Delta(\zeta)^{\frac{1}{2}} g(\zeta^{-1}) h(\zeta x)\, d\zeta \in C_0(G)$. Then

$$\epsilon(k, v)(x) = \int_{\Gamma_1} \int_{\Gamma_2} \delta_1(\xi)^{-\frac{1}{2}}\delta_2(\zeta)^{-\frac{1}{2}}\Delta(\zeta\xi)^{\frac{1}{2}} g(\zeta^{-1}) h(\zeta\xi x) L_\xi{}^{-1}v\, d\zeta d\xi \quad \text{so that}$$

$$\epsilon(k, v)\hat{}\,(\eta, x) = \int_{\Gamma_1} \int_{\Gamma_2} \delta_1(\xi)^{-\frac{1}{2}}\delta_2(\zeta\eta)^{-\frac{1}{2}}\Delta(\zeta\xi\eta)^{\frac{1}{2}} g(\zeta^{-1}) h(\zeta\xi\eta x) L_\xi{}^{-1}v\, d\zeta d\xi$$
$$= \int_{\Gamma_1} \int_{\Gamma_2} \delta_1(\xi)^{-\frac{1}{2}}\delta_2(\zeta\xi^{-1})^{-\frac{1}{2}}\Delta(\zeta)^{\frac{1}{2}} g(\xi\eta\zeta^{-1}) h(\zeta x) L_\xi{}^{-1}v\, d\zeta d\xi$$
$$= \int_{\Gamma_2} \delta_2(\zeta)^{-\frac{1}{2}}\Delta(\zeta)^{\frac{1}{2}} h(\zeta x) [\int_{\Gamma_1} \delta_1(\xi)^{-\frac{1}{2}}\delta_2(\xi)^{\frac{1}{2}} g(\xi\eta\zeta^{-1}) L_\xi{}^{-1}v\, d\xi]\, d\zeta.$$

Letting $\epsilon_1$, $\epsilon_2$ be the $\epsilon$-maps for the inductions from $\Gamma_1$ to $\Gamma_2$, $\Gamma_2$ to $G$ respectively, we see that $\epsilon(k,v)\hat{} = \epsilon_2(h,\epsilon_1(g,v))$. By Lemma 2, the set $\epsilon_2(C_0(G)\times\epsilon_1(C_0(\Gamma_2)\times\mathfrak{V}))$ is total in $\mathfrak{F}^{(2)}$, proving our assertion. Finally, we see that the map $f\to\hat{f}$ extends to a unitary map of $\mathfrak{F}$ onto $\mathfrak{F}^{(2)}$ which sets up the desired equivalence.

**3. The representation $\partial V$.** Let $G$ be a Lie group and let $V$ be a unitary representation of $G$ on the Hilbert space $\mathcal{K}$. Let $X\in\mathfrak{g}$, the left invariant Lie algebra of $G$, and let $x(\cdot)$ be the one-parameter subgroup of $G$ such that $(Xf)(y) = D_t f(yx(t))|_{t=0}$ for all $f\in C_0^\infty(G)$. $dV(X)$ will denote the skew-adjoint infinitesimal generator of the one-parameter unitary group $V_{x(\cdot)}$ in $\mathcal{K}$. We will denote by $\mathcal{K}_\infty$ the largest submanifold of $\mathcal{K}$ contained in $\bigcap[\mathrm{dom}(dV(X)) : X\in\mathfrak{g}]$ and invariant under $dV(\mathfrak{g})$. $\mathcal{K}_\infty$ is invariant under $V$ because $dV(X)V_y = V_y dV((\mathrm{ad}\,y^{-1})X)$, $X\in\mathfrak{g}$, $y\in G$.

Let $\partial V$ be the restriction of $dV$ to $\mathcal{K}_\infty$.

LEMMA 3. *$\mathcal{K}_\infty$ is dense in $\mathcal{K}$ and $\partial V$ is a representation of $\mathfrak{g}$ in the Lie algebra of all skew symmetric linear transformations of $\mathcal{K}_\infty$ into itself.*

*Proof.* Let $\mathfrak{D}$ be the linear space spanned by all vectors of the form

$$\int f(x)V_x^{-1}v\,dx \text{ for } f\in C_0^\infty(G),\ v\in\mathcal{K}.$$

Gårding has shown ([5]) that $\mathfrak{D}\subseteq\bigcap[\mathrm{dom}(dV(X)) : X\in\mathfrak{g}]$, is dense in $\mathcal{K}$, and is invariant under all $dV(X)$, and that the restriction $\delta V$ of $dV$ to $\mathfrak{D}$ is a representation of $\mathfrak{g}$. Therefore $\mathfrak{D}\subseteq\mathcal{K}_\infty$ and $\mathcal{K}_\infty$ is dense in $\mathcal{K}$. Segal has shown ([13]) that $\delta V(X)$ is essentially skew-adjoint, $X\in\mathfrak{g}$.

Let $X,Y\in\mathfrak{g}$. Then $\partial V(X)+\partial V(Y)\supseteq\delta(X+Y)$ and $[\partial V(X),\partial V(Y)]\supseteq\delta V([X,Y])$. Since the left hand members are skew-symmetric, they must be contained in the closures of the respective right hand members. Restricting these closures to $\mathcal{K}_\infty$, we have our result.

Let $\mathfrak{E}$ be the enveloping algebra of the complexification of $\mathfrak{g}$. Lemma 3 allows us to extend $\partial V$ to be a representation of $\mathfrak{E}$ in the algebra of endomorphisms of $\mathcal{K}_\infty$. If $+$ is the unique involutory conjugate linear anti-automorphism of $\mathfrak{E}$ such that $X^+ = -X$ for $X\in\mathfrak{g}$, it is easy to see that $\partial V(X^+)\subseteq\partial V(X)^*$ for $X\in\mathfrak{E}$ (cf. [12]).

LEMMA 4. *Let $V^1$ and $V^2$ be unitary representations of $G$ on $\mathcal{K}^1$ and $\mathcal{K}^2$ respectively. Let $A\in\mathfrak{R}(V^1,V^2)$, the space of operators intertwining $V^1$ and $V^2$* ([10], § 8). *Then $A\mathcal{K}^1_\infty\subseteq\mathcal{K}^2_\infty$. Moreover $A\partial V^1(X)\subseteq\partial V^2(X)A$ for $X\in\mathfrak{E}$.*

*Proof.* Let $X \in \mathfrak{g}$ and let $x(\cdot)$ be the one-parameter subgroup of $G$ underlying $X$. We have $V^2_{x(t)}A = AV^1_{x(t)}$, $t \in \boldsymbol{R}$. Hence, if $v \in \operatorname{dom}(dV^1(X))$, $Av \in \operatorname{dom}(dV^2(X))$ and $dV^2(X)Av = AdV^1(X)v$; i.e., $dV^2(X)A \supseteq AdV^1(X)$. We conclude that $A\mathcal{K}^1_\infty \subseteq [\operatorname{dom}(dV^2(X)).\ X \in \mathfrak{g}]$ and that $A\mathcal{K}^1_\infty$ is invariant under $dV^2(\mathfrak{g})$. Therefore $A\mathcal{K}^1_\infty \subseteq \mathcal{K}^2_\infty$. It follows that $A\partial V^1(X) \subseteq \partial V^2(X)A$ for $X \in \mathfrak{g}$. The same statement for $X \in \mathfrak{G}$ is immediate.

In the next two lemmas, we return to the situation of Section 2. $G$ is a Lie group. We shall denote by $\mathcal{F}_0^\infty$ the space of all infinitely strongly differentiable functions in $\mathcal{F}_0$.

LEMMA 5. $\mathcal{F}_0^\infty \subseteq \mathcal{H}_\infty$. *Moreover* $\partial U^L(X)f = Xf$ *for all* $X \in \mathfrak{G}$, $f \in \mathcal{F}_0^\infty$.

*Proof.* Let $X \in \mathfrak{g}$ and let $x(\cdot)$ be the one-parameter subgroup of $G$ underlying $X$. Let $f \in \mathcal{F}_0^\infty$. There is a compact subset $K$ of $M$ such that $\pi^{-1}(K)$ contains the supports of $Xf$ and $R_{x(t)}f$, $|t| \leqq 1$. Choose $h \in C_0^+(G)$ such that $\tau h = 1$ on $K$. Then

$$\| t^{-1}(U^L_{x(t)}f - f) - Xf \|^2 = \int h(y) \| t^{-1}(f(yx(t)) - f(y)) - (Xf)(y) \|^2 \, dy$$

for $0 < |t| \leqq 1$. But

$$\begin{aligned} \| t^{-1}(f(yx(t)) - f(y)) - (Xf)(y) \|^2 &\leqq 2 \, \| t^{-1}(f(yx(t)) - f(y)) \|^2 \\ + 2 \, \| (Xf)(y) \|^2 &\leqq 2 \sup[\| (Xf)(yx(t')) \|^2 : |t'| \leqq 1] \\ + 2 \, \| (Xf)(y) \|^2 &\leqq 4 \, \| (Xf)(y) \|_G^2. \end{aligned}$$

The bounded convergence theorem applies to show that $dU^L(X)f = Xf$. From this the lemma follows exactly as in Lemma 4.

LEMMA 6. *Let* $f \in C_0^\infty(G)$, $v \in \mathcal{V}$. *Then* $\epsilon(f, v) \in \mathcal{H}_\infty$. *Moreover* $\partial U^L(X)\epsilon(f, v) = \epsilon(Xf, v)$ *for all* $X \in \mathfrak{G}$.

*Proof.* Let $X \in \mathfrak{g}$ and let $x(\cdot)$ be the one-parameter subgroup of $G$ underlying $X$. There is a compact subset $K$ of $G$ containing the supports of $Xf$ and $R_{x(t)}f$, $|t| \leqq 1$. Then

$$\begin{aligned} &\| t^{-1}(U_{x(t)}\epsilon(f, v) - \epsilon(f, v)) - \epsilon(Xf, v) \| \\ &\quad = \| \epsilon(t^{-1}(R_{x(t)}f - f) - Xf, v) \| \leqq \Lambda_k \, \| t^{-1}(R_{x(t)}f - f) - Xf \|_G \, \| v \| \end{aligned}$$

if $0 < |t| \leqq 1$, using Lemma 2 and the fact that $R_y \circ \epsilon = \epsilon \circ (R_y \times I)$, $y \in G$. We conclude that $dU^L(X)\epsilon(f, v) = \epsilon(Xf, v)$, and the lemma follows as in Lemma 4.

COROLLARY. *Let $f \in C_0^\infty(G)$, $X \in \mathfrak{G}$. Then $v \to \partial U^L(X)\epsilon(f,v)$ is a bounded linear map from $\mathfrak{V}$ to $\mathfrak{H}$.*

*Proof.* From Lemmas 2 and 6.

**4. Certain elliptic operators.** Let $\mathfrak{V}_i$ be a finite dimensional Hilbert space and $\Gamma_i = (X_i, \mathfrak{F}_i, \mu_i)$ be a measure space, $i = 1, 2$ ($\mu_i$ is a measure defined on the $\sigma$-field $\mathfrak{F}_i$ of subsets of $X_i$). $\mathfrak{H}_i = L_2(\Gamma_i; \mathfrak{V}_i)$ will then be the Hilbert space of all measurable functions $f$ from $X_i$ to $\mathfrak{V}_i$ such that $\| f \| = (\int \| f(x) \|^2 d\mu_i(x))^{\frac{1}{2}} < \infty$ (modulo those for which $\| f \| = 0$). $\mathcal{L}(\mathfrak{V}_1; \mathfrak{V}_2)$ is also a Hilbert space under the norm $\| T \| = \text{trace } (T^*T)^{\frac{1}{2}}$ so that we can form the Hilbert space $\mathfrak{H}_{12} = L_2(\Gamma_1 \times \Gamma_2; \mathcal{L}(\mathfrak{V}_1; \mathfrak{V}_2))$. If $\sigma \in \mathfrak{H}_{12}$ and $f \in \mathfrak{H}_2$, then $\sigma(\cdot, f) = \int \sigma(\cdot, x) f(x) d\mu(x)$ is in $\mathfrak{H}_1$. Therefore the map $f \to \sigma(\cdot, f)$ is in $\mathcal{L}(\mathfrak{H}_2; \mathfrak{H}_1)$. It is easily seen that its (operator) norm is $\leqq \| \sigma \|$. Letting $\mathfrak{H}_{21} = L_2(\Gamma_2 \times \Gamma_1; \mathcal{L}(\mathfrak{V}_2; \mathfrak{V}_1))$, we easily see that the adjoint of $f \to \sigma(\cdot, f)$, $\sigma \in \mathfrak{H}_{12}$, is the map $g \to \sigma^*(\cdot, g)$ where $\sigma^* \in \mathfrak{H}_{21}$ is defined by $\sigma^*(x, y) = \sigma(y, x)^*$ for $(x, y) \in X_2 \times X_1$. The next lemma deals with certain special kernels $\sigma$.

LEMMA 7. *Let $X_i$ be an open subset of $\mathbf{R}^n$, $\mathfrak{F}_i$ the Borel field of $X_i$, and $\mu_i$ Lebesgue measure on $\mathfrak{F}_i$, $i = 1, 2$. Let $\sigma \in \mathfrak{H}_{12}$ have the following properties: (1) there is an $h \in L_2(\mathbf{R}^n)$ such that $\| \sigma(x, y) \| \leqq h(x - y)$ for $(x, y) \in X_1 \times X_2$; (2) there is a null set $N \subseteq \mathbf{R}^n$ such that $\sigma$ is continuous on $[(x, y) \in X_1 \times X_2 \colon x - y \notin N]$. Then, for all $f \in \mathfrak{H}_2$, $\sigma(\cdot, f) \in C(X_1; \mathfrak{V}_1)$ and $\| \sigma(\cdot, f) \|_{X_1} \leqq \| h \| \, \| f \|$.*

*Proof.*

$$
\begin{aligned}
\| \sigma(x, f) \| &\leqq \int_{X_2} \| \sigma(x, y) \| \, \| f(y) \| \, dy \\
&\leqq \left( \int \| h(x - y) \|^2 \right)^{\frac{1}{2}} \left( \int_{X_2} \| f(y) \|^2 dy \right)^{\frac{1}{2}} = \| h \| \, \| f \|.
\end{aligned}
$$

Now if $f \in C_0(X_2; \mathfrak{V}_2)$, we have $\sigma(x, f) = \int \sigma(x, x + y) f(x + y) dy$. Except for $y \in N$, the function $x \to \sigma(x, x + y) f(x + y)$ is continuous from $X_1$ to $\mathfrak{V}_1$. Moreover, if $B$ is bounded open subset of $X_1$ and $S$ is the support of $f$, then $\| \sigma(x, x + y) f(x + y) \| \leqq h(y) \| f \|_{X_2} \chi_{S+(-B)}(y) \in L_1(\mathbf{R}^n)$ for all $x \in B$. Hence the bounded convergence theorem applies to give the continuity of $\sigma(\cdot, f)$ when $f \in C_0(X_2; \mathfrak{V}_2)$. The same for general $f$ follows from the estimate

$\| \sigma(\cdot, f) \|_{X_1} \leqq \| h \| \, \| f \|$ for $f \in \mathfrak{H}_2$ and from the fact that $C_0(X_2; \mathfrak{V}_2)$ is dense in $\mathfrak{H}_2$.

*Remark.* If $\sigma$ is a kernel of the above type, so is $\sigma^*$ (with interchange of $X_1$ and $X_2$, etc.).

We shall apply this lemma in proving a result on analytic elliptic linear differential operators with coefficients in $\mathcal{L}(\mathfrak{V}; \mathfrak{V})$, $\mathfrak{V}$ a finite dimensional Hilbert space. Let $\boldsymbol{J}_+$ denote the non-negative integers. If $\zeta = (\zeta_1, \cdots, \zeta_n) \in \boldsymbol{R}^n$ and $s = (s_1, \cdots, s_n) \in \boldsymbol{J}_+{}^n$, we put $\zeta^s = \prod \zeta_i{}^{s_i}$. Moreover, if $|s| = \sum s_i$, then $\partial^{|s|}/\partial x^s$ will denote the operator $\partial^{s_1}/\partial x_1{}^{s_1} \cdots \partial^{s_n}/\partial x_n{}^{s_n}$ on all complex valued $C^{|s|}$ functions defined on open subsets of $\boldsymbol{R}^n$. Let $O$ be an open subset of $R^n$ and let $m \in \boldsymbol{J}_+$. Let $A_s(\cdot) \in C^\omega(O; \mathcal{L}(\mathfrak{V}; \mathfrak{V}))$, $|s| \leqq m$. The analytic linear differential operator $L = \sum_{|s| \leqq m} A_s(\cdot) \partial^{|s|}/\partial x^s$ in $O$ of order $m$ will be called *elliptic* if, for every $x \in O$, $Q(x; \zeta) = \sum_{|s| = m} A_s(x) \zeta^s$ is singular only if $\zeta = 0$. Although this definition of ellipticity is more restrictive than that given in [8], it is independent of basis in $\mathfrak{V}$ and suffices for our purposes.

PROPOSITION 3. *Let $L$ be an analytic elliptic linear differential operator of order $m$ with coefficients in $\mathcal{L}(\mathfrak{V}; \mathfrak{V})$ and defined in the open set $O$ of $\boldsymbol{R}^n$. Suppose $m > n/2$. Let $V$ be an isometry of $L_2(O; \mathfrak{V})$ into a Hilbert space $\mathcal{K}$. Let $T$ be a densely defined operator in $\mathcal{K}$ such that $VC_0{}^\infty(O; \mathfrak{V}) \subseteq \mathrm{dom}(T)$ and $TVf = VLf$ for $f \in C_0{}^\infty(O; \mathfrak{V})$. Then:* (1) $V^*(\mathrm{dom}(T^*)) \subseteq C(O; \mathfrak{V})$; (2) *for every compact subset $K$ of $O$ there is a constant $C_K$ such that $\| V^* v \|_K \leqq C_K(\| V^* T^* v \| + \| V^* v \|)$ for all $v \in \mathrm{dom}(T^*)$.*

*Proof.* We follow closely the proof of Lemma 2.1 in [6] (cf. [15], Theorem 1). Let $x_0 \in O$. Let $O_1$ be a bounded open neighborhood of $x_0$ such that $L$ has a fundamental solution $H$ defined on $O_1 \times O_1$ with the properties (see [8], Chapter III): (a) $H \in C^\omega(D; \mathcal{L}(\mathfrak{V}; \mathfrak{V}))$, where

$$D = [(x, y) \in O_1 \times . O_1 : x \neq y].$$

(b) $\| H(x, y) \| = O(\| x - y \|^{m-n-\epsilon})$ for all $\epsilon > 0$, where $\| x \|$ is the Euclidean norm of $x$ in $\boldsymbol{R}^n$; (c) for all $y \in O_1$, $LH(\cdot, y) = 0$ on $O_1$ except at $y$; (d) $g = L \int H(\cdot, y) g(y) dy$ for all $g \in C_0{}^\infty(O_1; \mathfrak{V})$. From [8], Chapter VII, we know that $\int H(\cdot, y) g(y) dy \in C^\infty(O_1; \mathfrak{V})$ for all $g \in C_0{}^\infty(O_1; \mathfrak{V})$. Choose $k \in C_0{}^\infty(O_1)$ such that $k = 1$ in an open neighborhood $O_2$ of $x_0$. Exactly as in [6], we see that

$$g = L \int k(\cdot) H(\cdot, y) g(y) dy + \int L[(1 - k(\cdot) H(\cdot, y))] g(y) dy$$

on $O_1$ for all $g \in C_0^\infty(O_2; \mathfrak{V})$.

For $(x,y) \in O \times O_2$, set $\xi(x,y) = k(x)H(x,y)$ or 0 and set $\eta(x,y) = L_x[(1 - k(x))H(x,y)]$ or 0, according as $x \in O_1$ or not. Properties (a) and (c) of $H$ imply that $\eta$ is a bounded continuous function on $O \times O_2$ and so is a kernel of the type considered in Lemma 7. So is $\xi$ by virtue of properties (a) and (b) of $H$ and the choice of $m$. Note also that $\xi(\cdot, g) \in C_0^\infty(O; \mathfrak{V})$ whenever $g \in C_0^\infty(O_2; \mathfrak{V})$.

Let $v \in \operatorname{dom}(T^*)$. Then, for all $g \in C_0^\infty(O_2; \mathfrak{V})$, we have

$$(V^*v, g) = (V^*v, L\xi(\cdot, g)) + (V^*v, \eta(\cdot, g)).$$

The first term

$$= (v, VL\xi(\cdot, g)) = (v, TV\xi(\cdot, g)) = (V^*T^*v, \xi(\cdot, g) = (\xi^*(\cdot, V^*T^*v), g),$$

while the second term $= (\eta^*(\cdot, V^*v), g)$. We conclude that $(V^*v)(x) = \xi^*(x, V^*T^*v) + \eta^*(x, V^*v)$ for almost all $x \in O_2$. From Lemma 7 and the remark following it we conclude the truth of our assertions in $O_2$. The general statements are immediate because of the arbitrariness of $x_0$.

We now apply Proposition 3 to prove the key result needed for our intertwining number theorems. A member of $E$ will be called *elliptic* if it is elliptic regarded as a left invariant (analytic) linear differential operator. We use the notation of Sections 2 and 3.

THEOREM 2. *Suppose* $\dim \mathfrak{V} < \infty$. *Then* $\mathcal{H}_\infty \subseteq C(G; \mathfrak{V})$. *Suppose, moreover, that* $X_0$ *is an elliptic element of* $\mathfrak{E}$ *of order* $m > n/2$, *where* $n = \dim M$. *Then for every compact subset* $K$ *of* $G$ *there is a constant* $C_K$ *such that* $\| g \|_K \leqq C_K(\| \partial U^L(X_0) g \| + \| g \|)$ *for all* $g \in \mathcal{H}_\infty$.

*Proof.* Clearly there is nothing to prove if $n = 0$ (in fact, in this case, the assumptions of ellipticity and order are unnecessary—we may take $X_0 = 0$). So assume $n \geqq 1$. Let $p_0 \in M$ and let $\phi$ be a $C^\omega$ diffeomorphism from the open unit sphere $O_1$ of $\boldsymbol{R}^n$ to an open neighborhood of $p_0$ in $M$ over which there is defined a $C^\omega$ cross-section $\alpha$ into $G$. Set $\beta = \alpha \circ \phi$ and $\psi = \phi^{-1} \circ \pi$. The map $\omega$ of $\Gamma \times O_1$ onto $\psi^{-1}(O_1)$ defined by $\omega(\xi, p) = \xi\beta(p)$ is a $C^\omega$ diffeomorphism. Let $\mu$ be right Haar measure on $G$ and set $\nu = \mu \circ \omega$, a Radon measure in $\Gamma \times O_1$. Let $\lambda$ be the Cartesian product of right Haar measure in $\Gamma$ and Lebesgue measure in $O_1$. If $\gamma$ is the density of $\nu$ with respect to $\lambda$, clearly $0 < \gamma \in C^\omega(\Gamma \times O_1)$. For $\xi_1 \in \Gamma$ and $(\xi, p) \in \Gamma \times O_1$, set $\xi_1(\xi, p) = (\xi_1\xi, p)$.

Then it is clear that $\nu(\xi_1 S) = \Delta(\xi_1)\nu(S)$ and $\lambda(\xi_1 S) = \delta(\xi_1)\lambda(S)$ for all $\xi_1 \in \Gamma$ and all Borel $S \subseteq \Gamma \times O_1$. It follows that $\gamma(\xi_1\xi, p)\delta(\xi_1) = \gamma(\xi, p)\Delta(\xi_1)$ for $\xi, \xi_1 \in \Gamma$ and $p \in O_1$, so that $\gamma(e, p) = \delta(\xi)\Delta(\xi)^{-1}\gamma(\xi, p)$ for $(\xi, p) \in \Gamma \times O_1$.

Now let $O = (\frac{1}{2})O_1$ and choose $h \in C_0^+(\psi^{-1}(O_1))$ so that $\tau h = 1$ on $\phi(O)$. If $f \in L_2(O; \mathfrak{V})$, define $\hat{f}$ on $G$ by

$$\hat{f}(x) = \gamma(e, \psi(x))^{-\frac{1}{2}}\delta(x\beta(\psi(x))^{-1})^{\frac{1}{2}}\Delta(x\beta(\psi(x))^{-1})^{-\frac{1}{2}}L_{x\beta(\psi(x))^{-1}}f(\psi(x))$$

or 0 according as $x \in \psi^{-1}(O)$ or not. Clearly $\hat{f}$ satisfies properties (1) and (2) in the definition of $\mathfrak{F}^*$ (Section 2). Moreover

$$\int_O \| f(p) \|^2 \, dp$$
$$= \int_O \int_\Gamma \| f(p) \|^2 \gamma(e, p)^{-1}\delta(\xi)\Delta(\xi)^{-1}\gamma(\xi, p) h(\xi\beta(p)) \, d\xi dp$$

by the choice of $h$; and this, by the Fubini theorem and the definitions of $\gamma$ and $\hat{f}$, is

$$\int_{\Gamma \times O} \| \hat{f}(\xi\beta(p)) \|^2 h(\xi\beta(p)) \, d\nu(\xi, p) = \int_G \| \hat{f}(x) \|^2 h(x) \, dx.$$

This shows that $\| \hat{f}(\cdot) \|^2$ is integrable on compact subsets of $[x \colon h(x) > 0]$ and therefore of $\Gamma[x \colon h(x) > 0] \supseteq [x \colon (\tau h)(\pi(x)) > \frac{1}{2}] \supseteq [x \colon \hat{f}(x) \neq 0]$. Therefore $\hat{f}$ is locally integrable on $G$ and $\| \hat{f} \| = \| f \|_2$. Setting $Vf = \hat{f}$ for $f \in L_2(O; \mathfrak{V})$, we see that $V$ is an isometry of $L_2(O; \mathfrak{V})$ into $\mathfrak{H}$. A similar calculation shows that $(V^*g)(\cdot) = \lambda(e, \cdot)^{\frac{1}{2}} g(\beta(\cdot))$, a.e. in $O$, for $g \in \mathfrak{F}$. It is clear that $V$ maps $C_0^\infty(O; \mathfrak{V})$ onto the subspace of $\mathfrak{F}_0^\infty$ consisting of functions whose supports $\subseteq \psi^{-1}(O)$, a subspace $\subseteq \operatorname{dom}(\partial U^L(X))$ and invariant under $\partial U^L(X)$ for all $X \in \mathfrak{E}$.

If $f$ is a complex valued $C^\omega$ function defined on an open subset of $\Gamma \times O$, we set $\omega f = f \circ \omega^{-1}$. $X_0^+$ is elliptic. The analytic elliptic linear differential operator $\omega^{-1} \circ X_0^+ \circ \omega$ we write as $L_0 = \sum_{|r|+|s| \leq m} a_{rs}(\xi, p)\partial^{|r|+|s|}/\partial\xi^r\partial p^s$ in some neighborhood of $\{e\} \times O$ adapted to the Cartesian decomposition of $\Gamma \times O$. Then, if $f \in C_0^\infty(O; \mathfrak{V})$, we have

$$\begin{aligned}(V^*\partial U^L(X_0^+)Vf)(p) &= \gamma(e, p)^{\frac{1}{2}}(X_0^+\hat{f})(\beta(p)) \\ &= \sum_{|r|+|s| \leq m} \gamma(e, p)^{\frac{1}{2}} a_{rs}(e, p) \\ &\quad [(\partial^{|r|}/\partial\xi^r)(\delta(\xi)^{\frac{1}{2}}\Delta(\xi)^{-\frac{1}{2}}L_\xi)]_{\xi=e}(\partial^{|s|}/\partial p^s)(\gamma(e, p)^{-\frac{1}{2}}f(p)) \\ &= \sum_{|s| \leq m} A_s(p)(\partial^{|s|}/\partial p^s)f(p),\end{aligned}$$

where the $A_s \in C^\omega(O; \mathcal{L}(\mathfrak{V}; \mathfrak{V}))$. The operator $L = \sum_{|s| \leq m} A_s(\cdot)\partial^{|s|}/\partial p^s$ is *elliptic*. In fact, if $|s| = m$, $A_s(p) = a_{0s}(e,p)I$. Hence

$$Q(p;\zeta) = \sum_{|s|=m} A_s(p)\zeta^s = (\sum_{|s|=m} a_{0s}(e,p)\zeta^s)I,$$

which is singular only if $\zeta = 0$ by the ellipticity of $L_0$.

Proposition 3 says: (1) if $f \in \mathcal{H}_\infty$, then $V^*f \in C(O;\mathfrak{V})$; (2) for any compact subset $K_0$ of $O$ there exists a constant $C'_{K_0}$ such that $\| V^*f \|_{K_0} \leq C'_{K_0}$ $(\| \partial U^L(X_0)f \| + \| f \|)$ (we here make use of the fact that $\partial U^L(X_0) \subseteq \partial U^L(X_0{}^+)^*$). Thus $f(\beta(\cdot))$ is continuous on $O$ and we conclude that

$$x \to f(x) = \delta(x\beta(\psi(x))^{-1})^{\frac{1}{2}}\Delta(x\beta(\psi(x))^{-1})^{-\frac{1}{2}}L_{x\beta(\psi(x))^{-1}}f(\beta(\psi(x)))$$

is continuous on $\psi^{-1}(O)$. From the arbitrariness of $p_0$ and the existence of elliptic members of $\mathfrak{E}$ of arbitrarily high order (see, e.g., [15]) we conclude the truth of the first assertion of our theorem. Moreover, the second assertion holds for any compact $K \subseteq \psi^{-1}(O)$: we need only take $C_K = C'_{\psi(K)}$.

$$\mathrm{LUB}[\delta(x\beta(\psi(x))^{-1})^{-\frac{1}{2}}\Delta(x\beta(\psi(x))^{-1})^{-\frac{1}{2}}\gamma(e,\psi(x))^{-\frac{1}{2}} \| L_{x\beta(\psi(x))^{-1}} \| : x \in K].$$

The second assertion for general $K$ follows easily from this and the arbitrariness of $p_0$.

**5. An intertwining number theorem.** Let $\Gamma_1$ and $\Gamma_2$ be closed subgroups of the Lie group $G$ with modular functions $\delta_1$ and $\delta_2$ respectively. Let $L^{(i)}$ be a unitary representation of $\Gamma_i$ on the Hilbert space $\mathfrak{V}_i$, $i = 1, 2$. $U^{L^{(i)}}$ operates on $\mathcal{H}^{(i)}$. We shall assume that $\dim \mathfrak{V}_2 < \infty$. For each $A \in \mathfrak{R}(U^{L^{(1)}}, U^{L^{(2)}})$ we define a linear map $r_A$ from $C_0{}^\infty(G)$ to the linear maps of $\mathfrak{V}_1$ into $\mathfrak{V}_2$ as follows: for each $f \in C_0{}^\infty(G)$ and $v \in \mathfrak{V}_1$, set $r_A(f)v = (A\epsilon(f,v))(e)$. This definition makes sense: in fact, $\epsilon(f,v) \in \mathcal{H}^{(1)}{}_\infty$ by Lemma 6, which implies that $A\epsilon(f,v) \in \mathcal{H}^{(2)}{}_\infty$ by Lemma 4, so that $A\epsilon(f,v) \in C(G;\mathfrak{V})$ by Theorem 1 and the value of $A\epsilon(f,v)$ at $e$ is well determined. For $(\xi_1,\xi_2) \in \Gamma_1 \times \Gamma_2$ and for any function $f$ on $G$, we set $(\rho_{\xi_1,\xi_2}f) = f(\xi_1{}^{-1}x\xi_2)$, $x \in G$. We may now state our main theorem.

THEOREM 3. *Let $X_0$ be an elliptic element of $\mathfrak{E}$ of order $> \frac{1}{2}\dim(G/\Gamma_2)$. For $f \in C_0{}^\infty(G)$, set $\| f \|_{X_0} = \| X_0 f \|_G + \| f \|_G$. For each relatively compact open set $O$ of $G$, give $C_0{}^\infty(O)$ the topology induced by $\| \cdot \|_{X_0}$; give $C_0{}^\infty$ the corresponding inductive limit topology* [1]. *Let $\mathfrak{M}$ = the subspace of maps $Z \in \mathcal{L}(C_0{}^\infty(G); \mathcal{L}(\mathfrak{V}_1;\mathfrak{V}_2))$ such that*

$$Z(\rho_{\xi_1,\xi_2}f) = \delta_1(\xi_1)^{\frac{1}{2}}\delta_2(\xi_2)^{\frac{1}{2}}\Delta(\xi_1\xi_2{}^{-1})^{\frac{1}{2}}L^{(2)}{}_{\xi_2}Z(f)L^{(1)}{}_{\xi_1^{-1}}$$

*for all* $(\xi_1, \xi_2) \in \Gamma_1 \times \Gamma_2$ *and all* $f \in C_0^\infty(G)$. *Then the map* $A \to r_A$ *is a faithful linear map of* $\mathcal{R}(U^{L^{(1)}}, U^{L^{(2)}})$ *into* $\mathfrak{M}$.

*Proof.* Let $v \in \mathfrak{V}_1$, $A \in \mathcal{R}(U^{L^{(1)}}, U^{L^{(2)}})$, and $f \in C_0^\infty(O)$, $O$ a relatively compact open subset of $G$. By Theorem 2,

$$\| r_A(f)v \| \leqq C_{\{e\}}(\| \partial U^{L^{(2)}}(X_0)A\epsilon(f,v) \| + \| A\epsilon(f,v) \|).$$

But $\partial U^{L^{(2)}}(X_0)A\epsilon(f,v) = A\partial U^{L^{(1)}}(X_0)\epsilon(f,v) = A\epsilon(X_0 f, v)$ by Lemmas 4 and 6. Therefore

$$\begin{aligned} \| r_A(f)v \| &\leqq C_{\{e\}} \| A \| (\| \epsilon(X_0 f, v) \| + \| \epsilon(f,v) \|) \\ &\leqq C_{\{e\}} \| A \| \Lambda_{\bar{O}} \| v \| (\| X_0 f \|_G + \| f \|_G) = C_{\{e\}}\Lambda_{\bar{O}} \| A \| \| f \|_{X_0} \| v \|. \end{aligned}$$

This proves that $r_A \in \mathcal{L}(C_0^\infty(G); \mathcal{L}(\mathfrak{V}_1; \mathfrak{V}_2))$.

Suppose $r_A = 0$. Let $f \in C_0^\infty(G)$ and $v \in \mathfrak{V}_1$. For all $x \in G$ we have

$$(A\epsilon(f,v))(x) = (U^{L^{(2)}}{}_x A\epsilon(f,v))(e) = (AU^{L^{(1)}}{}_x \epsilon(f,v))(e) = r_A(R_x f)v = 0$$

because $U^{L^{(1)}}{}_x \circ \epsilon = \epsilon \circ (R_x \times I)$. Therefore $A$ is 0 on $\epsilon(C_0^\infty(G) \times \mathfrak{V}_1)$, a total subset of $\mathfrak{H}^{(1)}$ by the remark following Lemma 2.

To show that $r_A \in \mathfrak{M}$, let $f \in C_0^\infty(G)$, $v \in \mathfrak{V}_1$, and $(\xi_1, \xi_2) \in \Gamma_1 \times \Gamma_2$. Then $r_A(\rho_{\xi_1,\xi_2} f)v = r_A(R_{\xi_1}\rho_{\xi_2,e} f)v = (A\epsilon(\rho_{\xi_1,e} f, v))(\xi_2)$ exactly as above, and this $= \delta_2(\xi_2)^{\frac{1}{2}}\Delta(\xi_2)^{-\frac{1}{2}} L^{(2)}{}_{\xi_2} r_A(\rho_{\xi_1,e} f)$ by property (2) for functions of $\mathfrak{F}^*$. On the other hand,

$$\begin{aligned} \epsilon(\rho_{\xi_1,e} f, v)(x) &= \int_{\Gamma_1} \delta_1(\xi)^{-\frac{1}{2}}\Delta(\xi)^{\frac{1}{2}} f(\xi_1^{-1}\xi x) L^{(1)}{}_{\xi^{-1}} v \, d\xi \\ &= \int_{\Gamma_1} \delta_1(\xi_1\xi)^{-\frac{1}{2}}\Delta(\xi_1\xi)^{\frac{1}{2}} f(\xi x) L^{(1)}{}_{\xi^{-1}\xi_1^{-1}} v \delta_1(\xi_1) \, d\xi \\ &= \delta_1(\xi_1)^{\frac{1}{2}}\Delta(\xi_1)^{\frac{1}{2}}\epsilon(f, L^{(1)}{}_{\xi_1^{-1}} v)(x). \end{aligned}$$

Therefore $r_A \in \mathfrak{M}$.

If $V$ and $W$ are representations of $G$, $\dim \mathcal{R}(V, W)$ is called the *intertwining number* of $V$ and $W$ and is denoted by $I(V, W)$ ([10]). Theorem 3 has the following consequence.

COROLLARY. $I(U^{L^{(1)}}, U^{L^{(2)}}) \leqq \dim \mathfrak{M}$.

**6. Inducing from compact subgroups.** For notation, terminology, and facts about direct integrals of Hilbert spaces, we refer the reader to [4]. Let $X$ be a separable locally compact space and let $\mu$ be a positive Radon

measure for $X$. Let $t \to \mathfrak{H}(t)$ be a field of separable Hilbert spaces and let $G$ be a separable locally compact group. A *field of (unitary) representations of* $G$ is a map which assigns to each $t \in X$ a representation $U(t)$ of $G$ on $\mathfrak{H}(t)$. If $t \to \mathfrak{H}(t)$ is measurable, then $t \to U(t)$ is called *measurable* if $t \to U_x(t)$ is a measurable operator field for each $x \in G$. We know ([10], that $x \to \int^{\oplus} U_x(t)\,d\mu(t)$ is a representation of $G$ on $\int^{\oplus} \mathfrak{H}(t)\,d\mu(t)$, which we denote $\int^{\oplus} U(t)\,d\mu(t)$.

If $V$ and $W$ are representations of $G, J(V,W)$ will denote the dimension of the subspace of all Hilbert-Schmidt operators in $\mathfrak{R}(V,W)$ and will be called the *weak intertwining number* of $V$ and $W$ ([10]).

LEMMA 8. *Let* $(X,\mu)$ *be atom free, let* $t \to \mathfrak{H}(t)$ *be a measurable field of Hilbert spaces, and let* $t \to U(t)$ *be a measurable field of representations of the separable locally compact group* $G$. *Let* $V$ *be a representation of* $G$ *on the separable Hilbert space* $\mathfrak{K}$. *Then the* $Y = [t \in X : J(U(t),V) > 0]$ *is measurable. Moreover* $J(\int^{\oplus} U(t)\,d\mu(t), V) = 0$ *or* $\infty$ *according as* $\mu(Y) = 0$ *or* $> 0$.

*Proof.* Let $\mathfrak{K}'$ be the conjugate space of $\mathfrak{K}$ and let $V'$ be the representation of $G$ in $\mathfrak{K}'$ defined by $V'_x = {}^tV_{x^{-1}}$. The field of Hilbert spaces $t \to \mathfrak{H}(t) \otimes \mathfrak{K}'$ is measurable in a canonical fashion which sets up a unitary equivalence of $\int^{\oplus} (\mathfrak{H}(t) \otimes \mathfrak{K}')\,d\mu(t)$ with $(\int^{\oplus} \mathfrak{H}(t)\,d\mu(t)) \otimes \mathfrak{K}'$ ([4], Ch. II, § 1, Section 8). The field of representations $t \to U(t) \otimes V'$ is then measurable (ibid., § 2, Section 1) and the above unitary equivalence then give a unitary equivalence of $\int^{\oplus} (U(t) \otimes V')\,d\mu(t)$ with $(\int^{\oplus} U(t)\,d\mu(t)) \otimes V'$ (ibid., § 2, Section 6). Now $J(U(t),V) =$ number of times $U(t) \otimes V'$ discretely contains the one-dimensional identity representation of $G$ and similarly for $J(\int^{\oplus} U(t)\,d\mu(t), V)$ ([10], Lemma 8.1). Our lemma now follows from [10], Lemma 13.1.

We are now in a position to prove the main result of this section. We return to the situation of Section 5. Two representations are said to be *disjoint* if no non-0 subrepresentations of one is unitarily equivalent to any subrepresentation of the other.

THEOREM 4. *Suppose that* $\Gamma_1$ *and* $\Gamma_2$ *are compact and that* $\dim \mathfrak{V}_1$ *and*

$\dim \mathfrak{V}_2 < \infty$. *For each* $x \in G$, *let* ${}_xL^{(2)}$ *be the representation of* $x\Gamma_2x^{-1}$ *defined by* ${}_xL^{(2)}{}_\xi = L^{(2)}{}_{x^{-1}\xi x}$, $\xi \in x\Gamma_2x^{-1}$. *Suppose that, for locally almost all* $x \in G$, *the restrictions of* $L^{(1)}$ *and* ${}_xL^{(2)}$ *to* $\Gamma_1 \cap (x\Gamma_2x^{-1})$ *are disjoint. Then* $U^{L^{(1)}}$ *and* $U^{L^{(2)}}$ *are disjoint.*

*Proof.* Let $B$ be a positive definite bilinear form on $\mathfrak{g}$ invariant under the action of $\Gamma_2$ on $\mathfrak{g}$ via the adjoint representation of $G$ on $\mathfrak{g}$. Choose a basis $\{X_i\}$ of $\mathfrak{g}$, orthonormal with respect to $B$. Exactly as in the discussion of the Casimir operator in [14], Exposé n° 4, we see that $\Delta = \sum X_i^2$ is invariant under $\operatorname{ad}\Gamma_2$ (where the adjoint representation of $G$ has been extended to $\mathfrak{E}$). Moreover $1 - \Delta$ is an elliptic operator on $G$ such that $\int [(1-\Delta)f]\bar{f} \geqq \int |f|^2$ for all $f \in C_0^\infty(G)$ (see, for instance, [15]).

Now suppose $U^{L^{(1)}}$ and $U^{L^{(2)}}$ are not disjoint. Then $I(U^{L^{(1)}}, U^{L^{(2)}}) > 0$. Let $k$ be an integer $> (\frac{1}{4})\dim(G/\Gamma_2)$ and set $X_0 = (1-\Delta)^k$. Using $X_0$ to define $\mathfrak{M}$, we see from the Corollary to Theorem 3 that $\dim \mathfrak{M} > 0$. The map $(x, (\xi_1, \xi_2)) \to \xi_1^{-1}x\xi_2$ of $G \times (\Gamma_1 \times \Gamma_2)$ into $G$ gives the pair $(G, \Gamma_1 \times \Gamma_2)$ the structure of compact analytic transformation group. Hence there is a $Z \in \mathfrak{M}$ and a relatively compact open set $O$ invariant under the action of $\Gamma_1 \times \Gamma_2$ such that $Z \mid C_0^\infty(O) \neq 0$. Let $l$ be an integer $> k + (\frac{1}{4})\dim(G)$ and let $S$ be the linear operator in $L_2(O)$ with domain $C_0^\infty(O)$ defined by $Sf = (1-\Delta)^l f$, $f \in C_0^\infty(O)$. $S \geqq I$ on $C_0^\infty(O)$ and so is univalent. We define a representation $P$ of $\Gamma_1 \times \Gamma_2$ on $L_2(O)$ by setting $P_{\xi_1,\xi_2}f = \rho_{\xi_1,\xi_2}f$, $f \in L_2(O)$. Now $\Delta \circ \rho = \rho \circ \Delta$ because $\Delta$ is left-invariant and is invariant under $\operatorname{ad}\Gamma_2$. Therefore $SP_{\xi_1,\xi_2} = P_{\xi_1,\xi_2}S$ so that $S^{-1}P_{\xi_1,\xi_2} = P_{\xi_1,\xi_2}S^{-1}$ for all $(\xi_1, \xi_2) \in \Gamma_1 \times \Gamma_2$. Setting $T = ZS^{-1}$, we have a mapping T from a submanifold of $L_2(O)$ to $\mathfrak{L}(\mathfrak{V}_1; \mathfrak{V}_2)$ such that $TP_{\xi_1,\xi_2}f = L^{(2)}{}_{\xi_2}(Tf)L^{(1)}{}_{\xi_1^{-1}}$ for all $f \in \operatorname{dom}(T)$. If we identify $\mathfrak{L}(\mathfrak{V}_1; \mathfrak{V}_2)$ with $\mathfrak{V}_2 \otimes \mathfrak{V}_1'$, we therefore obtain the relation $TP_{\xi_1,\xi_2} = V_{\xi_1,\xi_2}T$, $(\xi_1, \xi_2) \in \Gamma_1 \times \Gamma_2$, where $V_{\xi_1,\xi_2} = L^{(2)}{}_{\xi_2} \otimes L^{(1)}{}_{\xi_1}$ (see [10], § 5). Let $S_1 = (1-\Delta)^l \mid C_0^\infty(G)$ and $S_2 = (1-\Delta)^{l-k} \mid C_0^\infty(G)$, operators which are symmetric and $\geqq I$ in $L_2(G)$. Applying Proposition 3 to $S_1$ and $S_2$, we obtain constants $C_1$ and $C_2$ such that $\| f \|_{\bar{o}} \leqq C_1 \| S_1 f \|$ and $\| f \|_{\bar{o}} \leqq C_2 \| S_2 f \|$ for $f \in C_0^\infty(G)$. Since $S_1 = S_2X_0 = S$ on $C_0^\infty(O)$, it follows that $\| f \|_{X_0} = \| X_0 f \|_{\bar{o}} + \| f \|_{\bar{o}} \leqq (C_1 + C_2) \| Sf \|$ for all $f \in C_0^\infty(O)$. We conclude that $T$ is continuous. Therefore, since $\operatorname{dom}(T)$ is invariant under $P$, the unique bounded extension $T^c$ of $T$ to $L_2(O)$ which vanishes on $\operatorname{dom}(T)^\perp$ belongs to $\mathfrak{R}(P, V) > 0$, and $I(P, V) > 0$. Since $\dim(\mathfrak{V}_2 \otimes \mathfrak{V}_2') < \infty$, it follows that $J(P, V) > 0$.

$O$, being open and relatively compact, is a separable locally compact

space. Let $\mathfrak{X}$ be the space of orbits of $O$ under the action of $\Gamma_1 \times \Gamma_2$. $\mathfrak{X}$ is also a separable locally compact space in the usual topology. We put a Radon measure $\nu_t$ on each (compact separable space) $t \in \mathfrak{X}$ as follows: pick a point $x_0 \in t$ and then set $\int f(x) d\nu_t(x) = \int_{\Gamma_1 \times \Gamma_2} f(\xi_1^{-1} x_0 \xi_2) d(\xi_1, \xi_2)$ for all $f \in C(t)$, where Haar measure on $\Gamma_1 \times \Gamma_2$ is normalized so that $\int 1 d(\xi_1, \xi_2) = 1$. The two-sided invariance of Haar measure on compact kroups shows that $\nu_t$ is independent of the choice of $x_0 \in t$. We also define a measure $\mu$ on $\mathfrak{X}$ by setting $\int_{\mathfrak{X}} f(t) d\mu(t) = \int_O (f \circ \pi)(x) dx$ for each $f \in C_0(X)$, where $\pi$ is the usual projection of $O$ on $\mathfrak{X}$. Let $\mathfrak{H}(t) = L_2(t, \nu_t)$, a separable Hilbert space for each $t \in \mathfrak{X}$. Each $f \in C_0(O)$ defines a vector field $\theta f$, where $(\theta f)(t) = f \mid t$. If $f, g \in C_0(O)$, then

$$((\theta f)\pi(x)), (\theta g)(\pi(x))) = \int f(\xi_1^{-1} x \xi_2) \bar{g}(\xi_1^{-1} x \xi_2) d(\xi_1, \xi_2),$$

whence $t \to ((\theta f)(t), (\theta g)(t))$ is continuous on $\mathfrak{X}$. Moreover, if $\{f_n\}$ is a multiplicatively closed sequence of real functions in $C_0(O)$ which separates the points of $O$, the Stone-Weierstrass theorem implies that the sequence $\{(\theta f_n)(t)\}$ is total in $\mathfrak{H}(t)$ for all $x \in \mathfrak{X}$. Thus the field of Hilbert spaces $t \to \mathfrak{H}(t)$ has a unique measurable structure such that all vector fields in $G_0 = \theta C_0(O)$ are measurable. Now

$$\begin{aligned} \| \theta f \|^2 &= \int_{\mathfrak{X}} \| (\theta f)(t) \|^2 d\mu(t) \\ &= \int_G \int_{\Gamma_1 \times \Gamma_2} | f(\xi_1^{-1} x \xi_2) |^2 d(\xi_1, \xi_2) dx = \int_G | f(x) |^2 dx = \| f \|^2, \end{aligned}$$

for all $f \in C_0(O)$, by the Fubini theorem and the invariance properties of Haar measure on $G$. Therefore we may identify $L_2(O)$ and $\int^{\oplus} \mathfrak{H}(t) d\mu(t)$. For each $t \in \mathfrak{X}$, we define a representation $P(t)$ of $\Gamma_1 \times \Gamma_2$ on $\mathfrak{H}(t)$ by setting $P_{\xi_1, \xi_2}(t) f = \rho_{\xi_1, \xi_2} f$ for $f \in \mathfrak{H}(t)$, $(\xi_1, \xi_2) \in \Gamma_1 \times \Gamma_2$. Since

$$P_{\xi_1, \xi_2}(t)(\theta f)(t) = (\theta P_{\xi_1, \xi_2} f)(t)$$

for each $(\xi_1, \xi_2) \in \Gamma_1 \times \Gamma_2$, $f \in C_0(G)$, and $t \in \mathfrak{X}$, we see that the field of representations $t \to P(t)$ is measurable and that $P$ may be identified with $\int^{\oplus} P(t) d\mu(t)$. From Lemma 8 and the fact that $V$ is of finite degree we

conclude that the $\mu$-measurable set $Y = [t \in \mathfrak{X}: I(P(t), V) > 0]$ has positive $\mu$-measure.

Fix $t \in \mathfrak{X}$ and pick $x_0 \in t$. The stationary subgroup of $x_0$ under the action of $\Gamma_1 \times \Gamma_2$ is $\Gamma_{x_0} = [(\xi, x_0^{-1}\xi x_0) : \xi \in \Gamma_1 \cap (x_0\Gamma_2 x_0^{-1})]$. If $f \in \mathfrak{H}(t)$, define $\chi_t f$ on $\Gamma_1 \times \Gamma_2$ by $(\chi_t f)(\xi_1, \xi_2) = f(\xi_1^{-1}x_0\xi_2)$. It is easy to see that $\chi_t$ identifies $P(t)$ with the representation of $\Gamma_1 \times \Gamma_2$ induced by the one-dimensional identity representation of $\Gamma_{x_0}$. Therefore, according to the Frobenius reciprocity theorem for compact groups ([16], pp. 82-83), $I(P(t), V) =$ number of times $V$ restricted to $\Gamma_{x_0}$ contains the one-dimensional identity representation. But this is just the weak intertwining number of the restrictions of $L^{(1)}$ and ${}_{x_0}L^{(2)}$ to $\Gamma_1 \cap (x_0\Gamma_2 x_0^{-1})$ ([10], Lemma 8.1). We conclude that the restrictions of $L^{(1)}$ and ${}_xL^{(2)}$ to $\Gamma_1 \cap (x\Gamma_2 x^{-1})$ are not disjoint whenever $x \in \pi^{-1}(Y)$, a measurable set of positive measure in the compact set $\bar{O}$. Our theorem is thereby demonstrated.

*Remark.* As usual (cf. [10]), the intertwining number of the restrictions of $L^{(1)}$ and $x^{L^{(2)}}$ to $\Gamma_1 \cap (x\Gamma_2 x_2^{-1})$ depends only on the $\Gamma_1 : \Gamma_2$ double coset $\Gamma_1 x \Gamma_2$ to which $x$ belongs.

**7. An example.** Let $G$ be the group of (possibly improper) rigid motions of the Euclidean plane $\Pi$, let $P \in \Pi$, and let $\Gamma$ be the group of (possibly improper) rotations of $\Pi$ about $P$. Let $L^{(+)}$ (resp. $L^{(-)}$) be the 1-dimenstional identity representation of $\Gamma$ (resp. the 1-dimensional representation of $\Gamma$ which assigns to $\xi \in \Gamma$ the value of $+1$ or $-1$ as $\xi$ is proper or improper). If $T$ is the subgroup of translations of $\Pi$, $G = T\Gamma$ shows that each $\Gamma : \Gamma$ double coset of $G$ contains a member of $T$. Let $e \neq x \in T$. Then $x\Gamma x^{-1}$ is the group of (possibly improper) rotations about $xP$, so that $\Gamma \cap (x\Gamma x^{-1})$ is the group consisting of $e$ and reflection in the line through $P$ and $xP$. Therefore the restrictions of $L^{(-)}$ and ${}_xL^{(+)}$ to $\Gamma \cap (x\Gamma x^{-1})$ are disjoint, and by the remark following Theorem 4, this holds for all $x \notin \Gamma$. The same holds if $x \in \Gamma$, for then $\Gamma \cap (x\Gamma x^{-1}) = \Gamma$. From Theorem 4 we conclude that $U^{L^{(+)}}$ and $U^{L^{(-)}}$ are disjoint.

We can derive the same result from Mackey's reciprocity theorem. Since $G$ is a regular semi-direct product of the abelian normal subgroup $T$ and $\Gamma$, we may apply the analysis of ([10], § 14) to obtain the following information about the irreducible representations of $G$. Let $\hat{T}$ be the character group of $T$ and let $\Gamma$ act on $\hat{T}$ according to the rule $(\chi\xi)(x) = \chi(\xi x \xi^{-1})$ for $\xi \in \Gamma$ $\chi \in \hat{T}$, $x \in T$. Let $\Omega$ be the space of orbits of $T$ under $\Gamma$ and, for each $\omega \in \Omega$, choose $\chi_\omega \in \omega$. Let $\Gamma_\omega$ be the stationary subgroup of $\chi_\omega$ in $\Gamma$. Then each

irreducible representation of $G$ is uniquely specified by a $\omega \in \Omega$ and an irreducible representation $M$ of $\Gamma_\omega$, and conversely. Calling this representation $V^{\omega,M}$, the restriction of $V^{\omega,M}$ to $\Gamma$ is the representation of $\Gamma$ induced by $M$. Now if $\omega = \{e\}$, then $\Gamma_\omega = \Gamma$; and if $\omega \neq \{e\}$, then $\Gamma_\omega$ is the subgroup consisting of the identity and the unique reflection in $\Gamma$ leaving $\chi_\omega$ fixed. Moreover if $M$ is an irreducible representation of $\Gamma_\omega$, we have that ${}_\Gamma U^M$ contains $L^{(\pm)}$ exactly as many times as the restriction of $L^{(\pm)}$ to $\Gamma_\omega$ contains $M$ (the Frobenius reciprocity theorem for compact groups). Thus in any case ${}_\Gamma U^M$ contains $L^{(+)}$ if and only if it does *not* contain $L^{(-)}$. Finally from results of Godement ([7], Theorems 5 and 7) and Kaplansky ([9], Theorem 7), we know that the regular representation of $G$ is of Type $I$. Therefore Mackey's reciprocity theorem ([11], Theorem 5.1) applies, and we deduce that $U^{L^{(+)}}$ and $U^{L^{(-)}}$ are disjoint.

It is instructive to note that in order to apply Mackey's reciprocity theorems to verify the conclusion of our Theorem 4 in any particular case one needs to know (1) how the regular representation of $G$ decomposes into factors and (2) how these factors, when restricted to $\Gamma_1$ and $\Gamma_2$, decompose into irreducible representations. Even in so simple a case as the motion group considered above, the machinery needed to dig out these facts is quite formidable. Thus the reciprocity theorem would not seem well suited in general to handle the disjointness question dealt with in Theorem 4.

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# ON CHOW VARIETIES OF MAXIMAL, TOTAL, REGULAR FAMILIES OF POSITIVE DIVISORS.* [1]

By J. P. Murre.

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**Introduction.** In this paper we study the Chow variety of a maximal, total, regular family of positive divisors on a non-singular projective variety $V$. An algebraic family $\mathfrak{U}$ of positive divisors on $V$ is called *maximal* if $\mathfrak{U}$ is not a proper subset of another algebraic family; it is called *total* if for every divisor $Y$ on $V$, algebraically equivalent to zero, and for an arbitrary fixed (i.e. independent of $Y$) $X_0 \in \mathfrak{U}$ there exists an $X \in \mathfrak{U}$ such that $Y \sim X - X_0$ ($\sim$ means linear equivalence); finally, it is called *regular* if for every pair $X, X' \in \mathfrak{U}$ we have $l(X) = l(X')$, where $l(X)$ denotes the dimension of the linear system determined by $X$. These definitions are introduced, and the existence of such families is proved, in [6, 7] (in [6, 7] such families are called maximal, regular, complete instead of maximal, total, regular in this order).

If $V$ is embedded in projective space $P^N$, then the Chow points are constructed by means of the hyperplanes in $P^N$, and therefore we must expect that there is some connection between the properties of the Chow variety $U$ of $\mathfrak{U}$ (for instance the non-singularity of $U$) and the way $V$ is embedded in $P^N$ (or to be more precise, the properties of the linear system of hyperplane sections on $V$). Our main purpose is to show that, under a mild condition on the embedding of $V$ in $P^N$, the Chow variety of a maximal, total, regular family is non-singular. As a preparation to this result we first study the Chow varieties of linear systems on $V$ and it turns out that, under the same condition on the embedding of $V$ in $P^N$, the Chow variety of a linear system is non-singular (Proposition 1).

As we have just mentioned we have to assume some properties for the linear system of hyperplane sections on $V$ in order to be able to prove the non-singularity of $U$; if these properties are fulfilled, we shall say that $V$ is *adaptable embedded* in $P^N$ (see the definition in Section 2). However, this is not a very serious restriction, for we shall see in Section 2 that the embedding

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$V'$ of $V$ by means of the hypersurface sections of degree $m$ $(m>1)$ in $P^N$ always has these properties (Lemma 5). This result can also be interpreted in the following way. If we construct the Chow points by using hypersurfaces of degree $m$ $(m>1)$ instead of hyperplanes, then the Chow varieties of maximal, total, regular families (and also of linear systems) are non-singular.

In Section 1 we study the degrees of Chow varieties; the Lemmas 1 and 2 are generalizations of results of Chow in [3].

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**1.** All varieties under consideration are assumed to be projective varieties; therefore the degree of a variety is defined. The ambient projective space of a variety is usually denoted by $P^N$ and $Z_0, Z_1, \cdots, Z_N$ are used as letters for $P^N$; the ambient projective space of the Chow variety of a system of positive cycles (of a certain dimension and a certain degree) in $P^N$ is usually denoted by $P^t$, and for $P^t$ we use the letters $Y_0, Y_1, \cdots, Y_t$. If $X$ is a positive cycle in $P^N$, then its Chow point is denoted by $\mathrm{Ch}(X)$; the degree of a variety $V$ is denoted by $\deg V$.

If $P^N$ is a projective space and if $u_{ij}$ $(j=0,\cdots,N; i=1,\cdots,n)$ is a set of elements of the universal domain $\Omega$, then we mean by the linear variety defined by the set of elements $(u_{ij})$ (or sometimes shortly $(u)$) the variety defined by the set of equations $\sum_{j=0}^{N} u_{ij}Z_j=0$ $(i=1,\cdots,n)$ in $P^N$.

First, we mention some facts which will be used frequently in the following. Let $V^n$ be a variety in $P^N$ defined over a field $k$ and of degree $h$. Let $(u_{ij,\sigma})$ $(j=0,\cdots,N; i=1,\cdots,n; \sigma=1,\cdots,r$, where $r$ is some integer) be a system of independent transcendental elements over $k$, and let $L_\sigma$ $(\sigma=1,\cdots,r)$ be the linear variety defined by the set $(u_{ij,\sigma})$ ($\sigma$ fixed). Then $L_\sigma \cdot V=\sum_{\alpha=1}^{h} Q_{\sigma\alpha}$, where all the $Q_{\sigma\alpha}$ are different from each other. If $K_\sigma$ denotes the field obtained by adjoining to $k$ all the $(u_{ij,\tau})$ with $\tau\neq\sigma$, then $Q_{\sigma\alpha}$ and $Q_{\sigma\beta}$ $(\beta\neq\alpha)$ are independent generic points of $V$ over $K_\sigma$ [5, Chap. VIII, Prop. 10]. Therefore, given two arbitrary sets of indices $(\alpha_1,\cdots,\alpha_r)$ and $(\beta_1,\cdots,\beta_r)$ with $1\leqq \alpha_i, \beta_j \leqq \mathrm{h}$, there exists a $k$-automorphism of the universal domain transforming $Q_{\sigma\alpha_\sigma}$ into $Q_{\sigma\beta_\sigma}$ $(\sigma=1,\cdots,r)$. Furthermore, let $U$ be the Chow variety of an algebraic system of positive divisors on $V$. Let $H_\sigma$ be the hyperplane in the ambient projective space $P^t$ of $U$ defined by the equation $\sum_\rho M_\rho(u_{ij,\sigma})Y_\rho=0$ (for some fixed $\sigma$), where the $M_\rho(U_{ij})$ range over the monomials of a suitable degree in the $U_{ij}$ (the

monomials occurring in the Chow forms of the divisors in the algebraic system associated with $U$). Such a hyperplane will be called a *hyperplane derived* from the set $(u_{ij,\sigma})$. Then we have by the properties of Chow coordinates (see [1]) for a divisor $X$ in the system that $\mathrm{Ch}(X) \in U \cap H_\sigma$ if and only if $X$ (or better the point set $|X|$) contains some point $Q_{\sigma\alpha}$.

LEMMA 1. *Let $V^n$ be a complete variety, non-singular in codimension* 1 *and of degree $h$. Let $U^r$ be the Chow variety of an algebraic family $\mathfrak{U}$ of positive divisors on $V$. Then* $\deg U \geqq h^r$.

*Proof.* Let $k$ be a field of definition for $V$ and $U$. Let $(u_{ij,\sigma})$ $(j=0,\cdots,N;\ i=1,\cdots,n;\ \sigma=1,\cdots,r)$ be a set of independent transcendental elements over $k$. If $L_\sigma$ $(\sigma=1,\cdots,r)$ is the linear variety in the ambient projective space $P^N$ of $V$ defined by the set $(u_{ij,\sigma})$ and if $L_\sigma\cdot V=\sum_{\alpha=1}^{h} Q_{\sigma\alpha}$, then we can apply to this intersection the remarks preceding the lemma. Let furthermore $H_\sigma$ $(\sigma=1,\cdots,r)$ be the hyperplanes, in the ambient space $P^t$ of $U$, derived from the sets $(u_{ij,\sigma})$.

First of all, we want to show that $U\cdot H_1\cdot\cdots\cdot H_r$ is defined. Clearly, it suffices to show that $U\cdot H_1$ is defined. Let $X$ be a divisor in $\mathfrak{U}$ and algebraic over $k$. Since every $Q_{1\alpha}$ is generic over $k$ on $V$, it follows that $X$ contains no point $Q_{1\alpha}$. Therefore (see the remarks preceding the lemma) $\mathrm{Ch}(X)\notin U\cap H_1$, i.e. $U \not\subset H_1$, i.e. $U\cdot H_1$ is defined.

Since $U\cap H_1\cap\cdots\cap H_r\neq\emptyset$ for dimension reasons and since $V$ is complete and non-singular in codimension 1, it follows that there exists a divisor $X$ in $\mathfrak{U}$ and a set of indices $(\alpha_1,\cdots,\alpha_r)$ such that $Q_{\sigma\alpha_\sigma}\in|X|$ for $\sigma=1,\cdots,r$; let us assume for convenience that $\alpha_\sigma=1$, i.e. that $Q_{\sigma 1}\in|X|$ for $\sigma=1,\cdots,r$. Given any two set of indices $(\alpha_1,\cdots,\alpha_r)$ and $(\beta_1,\cdots,\beta_r)$, we have seen above that there exists a $k$-automorphism of the universal domain transforming $Q_{\sigma\alpha_\sigma}$ into $Q_{\sigma\beta_\sigma}$. Since there are precisely $h^r$ such sets, it suffices, in order to prove the lemma, to show that, for some set of indices $(\alpha_1,\cdots,\alpha_r)$, there exists a divisor $X^*$ of $\mathfrak{U}$ such that $Q_{\sigma\alpha_\sigma}\in|X^*|$ for $\sigma=1,\cdots,r$ but $Q_{\sigma\beta}\notin|X^*|$ for all $\beta\neq\alpha_\sigma$ $(\sigma=1,\cdots,r)$. In order to see that, let $L_\sigma'$ $(\sigma=1,\cdots,r)$ be a linear variety in $P^N$ defined by a set $(u'_{ij,\sigma})$ which is such that:

1. $L_\sigma'$ goes through $Q_{\sigma 1}$ $(\sigma=1,\cdots,r)$,

2. over the field $K=k(Q_{11},Q_{21},\cdots,Q_{r1},\mathrm{Ch}(X))$ the set $(u')$ is a generic set which fulfills the condition 1.

Since the $Q_{\sigma 1}$ are $r$ independent generic points of $V$ over $k$, we clearly have over $k$ the generic specialization $(u)\to(u')$, where $(u)$ stands for the entire set $(\cdots,u_{ij,1},\cdots;\cdots;\cdots,u_{ij,r},\cdots)$, and $(u')$ similar. Let $L'_\sigma\cdot V=\sum_{\alpha=1}^{h} Q'_{\sigma\alpha}$

with, say, $Q'_{\sigma 1}=Q_{\sigma 1}$; from the remarks preceding the lemma it follows that $Q'_{\sigma\alpha}$ is a generic point of $V$ over $K$ for $\alpha\neq 1$. Extend the generic specialization $(u)\to(u')$ to a generic specialization $(u',Q',X)\to(u,Q,X^*)$ over $k$, where the set $(Q)$ contains all the $Q_{\sigma\alpha}$, and $(Q')$ similar. Let under this specialization $Q'_{\sigma 1}$ correspond with $Q_{\sigma\gamma_\sigma}$. Now $Q'_{\sigma 1}\in|X|$, but since $Q'_{\sigma\alpha}$ for $\alpha\neq 1$ is a generic point of $V$ over $K=K(\mathrm{Ch}(X))$, it follows that $Q'_{\sigma\alpha}\notin|X|$ for $\alpha\neq 1$ and $\sigma=1,\cdots,r$. Therefore it follows that $Q_{\sigma\gamma_\sigma}\in|X^*|$ but $Q_{\sigma\beta}\notin|X^*|$ for $\beta\neq\gamma_\sigma$ $(\sigma=1,\cdots,r)$. This completes the proof.

LEMMA 2. *Let $V^n$ be a complete variety, non-singular in codimension* 1 *and of degree $h$. Let $U^r$ be the Chow variety of a linear system $\mathcal{L}$ of divisors on $V$. Assume that $\mathcal{L}$ is without fixed component. Then* $\deg U=h^r$.

*Proof.* Let $k$ be a field of definition for $V$ and $U$ and such that the linear system has a module in the function field[1] which has a basis of functions defined over $k$. We have to consider two different cases.

*Case* 1. Suppose a generic element of $\mathcal{L}$ has no multiple components. Let $L_\sigma$ $(\sigma=1,\cdots,r)$ be the linear varieties in $P^N$ defined by the sets of elements $(u_{ij,\sigma})$ $(j=0,\cdots,N;1,\cdots,n)$, where all the $u_{ij,\sigma}$ are transcendental and independent from each other over $k$. Let $L_\sigma\cdot V=\sum_{\alpha=1}^{h}Q_{\sigma\alpha}$; then we have noted above that $Q_{\sigma\alpha}$ and $Q_{\sigma\beta}$ $(\alpha\neq\beta)$ are two independent generic points of $V$ over the field $K_\sigma$ (introduced above). Furthermore, for any set of indices $(\alpha_1,\cdots,\alpha_r)$ with $1\leqq\alpha_\sigma\leqq h$, we have that $Q_{1\alpha_1},\cdots,Q_{r\alpha_r}$ is a set of $r$ independent generic points of $V$ over $k$, and therefore there is precisely one element $X$ in the linear system going through $Q_{1\alpha_1},\cdots,Q_{r\alpha_r}$. If we denote this $X$ by $X_{\alpha_1\cdots\alpha_r}$, then $X_{\alpha_1\cdots\alpha_r}$ is rational over the field $K_{\alpha_1\cdots\alpha_r}=k(Q_{1\alpha_1},\cdots,Q_{r\alpha_r})$. Since, as we have seen above, $Q_{\sigma\beta}$ for $\beta\neq\alpha_\sigma$ is a generic point of $V$ over this field, it follows that $X_{\alpha_1\cdots\alpha_r}$ does not contain $Q_{\sigma\beta}$. Therefore $X_{\alpha_1\cdots\alpha_r}\neq X_{\beta_1\cdots\beta_r}$ if for some $\sigma$ the $\alpha_\sigma\neq\beta_\sigma$.

Next, let $H_\sigma$ $(\sigma=1,\cdots,r)$ be the hyperplane in the ambient projective space of $U$ derived from the set $(u_{ij,\sigma})$. Then we have, according to the remarks made above, that $\mathrm{Ch}(X_{\alpha_1\cdots\alpha_r})\in U\cap H_1\cap\cdots\cap H_r$ for every set of indices $(\alpha_1,\cdots,\alpha_r)$ and moreover these are the only points in this intersection. Since we obtain in this way precisely $h^r$ (different) points, it suffices, in order to complete the proof, to see that every point has multiplicity 1 in

[1] Let $\mathcal{L}$ be a linear system on $V$; let $\Omega$ be the universal domain and $\Omega(V)$ the function field of $V$. We shall say that a vector space $M\subset\Omega(V)$ over $\Omega$ is a module for $\mathcal{L}$ if there exists a (fixed) divisor $Y$ on $V$ such that $(f)+Y\in\mathcal{L}$ for all functions $f\in M$ and if conversely for every $X\in\mathcal{L}$ there exists an $f\in M$ such that $X=(f)+Y$. Such a module always exists; in particular, if we take an $X_0\in\mathcal{L}$, then the set $L(X_0)=\{f\,|\,f\in\Omega(V)$ such that $(f)=X-X_0$ with $X\in\mathcal{L}\}$ is a module for $\mathcal{L}$.

this intersection. From the fact that $X_{\alpha_1\cdots\alpha_r}$ contains the points $Q_{1\alpha_1},\cdots,Q_{r\alpha_r}$, which are $r$ independent generic points of $V$ over $k$, it follows easily that the point $\mathrm{Ch}(X_{\alpha_1\cdots\alpha_r})$ is generic on $U$ over $k$; therefore it is in particular simple. Since (as is easily seen) the $X_{\alpha_1\cdots\alpha_r}$ are conjugate to each other over the field obtained by adjoining the coefficients $u_{ij,\sigma}$ of the $L_\sigma$ to the ground field $k$, it suffices to consider only one of the $X_{\alpha_1\cdots\alpha_r}$, and let us write $X$ instead of $X_{\alpha_1\cdots\alpha_r}$. We have to show that $H_1\cdot\cdots\cdot H_r$ is transversal to the tangent space to $U$ at $\mathrm{Ch}(X)$. Since the $H_1,\cdots,H_r$ are $r$ independent generic derived hyperplanes over the ground field $k$, it suffices to show that there are some $r$ derived hyperplanes $H'_1,\cdots,H'_r$ such that $H'_1\cdot\cdots\cdot H'_r$ is defined and transversal with the tangent space to $U$ at $\mathrm{Ch}(X)$ (for then over the specialization $H_i\to H'_i$ over $k$ there is an $X_{\beta_1\cdots\beta_r}$ specializing to $X$ and hence $H_1\cdot\cdots\cdot H_r$ is transversal to the tangent space to $U$ at $\mathrm{Ch}(X_{\beta_1\cdots\beta_r})$, which is sufficient since $X_{\alpha_1\cdots\alpha_r}$ and $X_{\beta_1\cdots\beta_r}$ are conjugate). However, now we contend that the intersection of all the derived hyperplanes through $\mathrm{Ch}(X)$ is $\mathrm{Ch}(X)$ itself, which is clearly sufficient for the existence of $r$ derived hyperplanes $H'_1,\cdots,H'_r$ with the above mentioned property. In order to see that the contention is true let $\mathrm{Ch}(X)=(c_\lambda)$ and let $(d_\lambda)$ be an arbitrary point in the ambient space $P^t$ of $U$. Consider the Chow form $\sum c_\lambda M_\lambda(U)=F(U)$ and the form $\sum d_\lambda M_\lambda(U)=G(U)$. We must try to find a set $(u')$ such that $F(u')=0$ and $G(u')\neq 0$. However, if $(c_\lambda)\neq(d_\lambda)$, then the existence of such a set $(u')$ follows from the fact that $F(U)$ has no multiple factors since $X$ has no multiple components.

*Case* 2. Suppose a generic member $X$ (over $k$) of $\mathfrak{L}$ has multiple components. According to [9, Th. 1.6.3] we have $X=p^eX'$, where $X'$ has no multiple components and $p$ is the characteristic of the universal domain. In order not to interrupt the arguments later on we first state an auxiliary lemma.

LEMMA 3. *Let $U$ be a projective variety defined over a field $k$ of characteristic $p$ and let $P$ be a generic point of $U$ over $k$. Let $U^*$ be the locus of $P^{(p^e)}$ over the field $k^{(p^e)}$, where $P^{(p^e)}$ is the point obtained by raising the coordinates of $P$ to the $p^e$-th power. Then* $\deg U=\deg U^*$.

For the proof of Lemma 3 it suffices to remark that we obtain $U^*$ from $U$ by applying the Frobenius automorphism $\rho\to\rho^{p^e}$ to the universal domain and that by this automorphism hyperplanes go over in hyperplanes.

Returning to the proof of Case 2 of Lemma 2, take a fixed divisor $X_0$ in $\mathfrak{L}$ and let $k$ be an algebraically closed field over which $V$ and $U$ are defined, over which $X_0$ is rational and such that the function module $L(X_0)$ of $\mathfrak{L}$ has a base of functions defined over $k$. Let $X$ be a generic member of $\mathfrak{L}$ over $k$; then we have $X=p^eX'$ as we have seen above. Let $U'$ be the locus of the

$\mathrm{Ch}(X')$ over $k$. If $\mathrm{Ch}(Y') \in U'$, then $\mathrm{Ch}(p^e Y') \in U$, and conversely if $\mathrm{Ch}(Y) \in U$, then $Y = p^e Y'$ with a $Y'$ such that $\mathrm{Ch}(Y') \in U'$; in particular $X_0 = p^e X_0'$ with $\mathrm{Ch}(X_0') \in U'$. By [9, Prop. 1.6.4] we have $L(X_0) \subset (\Omega(V))^{(p^e)}$, where $\Omega$ is the universal domain and $\Omega(V)$ is the function field of $V$ over $\Omega$. If $\mathrm{Ch}(Y') \in U'$, then $p^e Y' = X_0 + (f)$ with $f \in L(X_0)$ and hence $Y' - X_0' = (f^{1/p^e})$ with $f^{1/p^e} \in \Omega(V)$, and hence $Y' \sim X_0'$. Moreover, it is then easily seen that $U'$ is the Chow variety of a linear system $\mathcal{L}'$ without fixed components, a generic element $X'$ of which has no multiple components. Let $P^{t'}$ be the ambient space of $U'$. Let $(c_\lambda) = \mathrm{Ch}(X')$ and $(d_\mu) = \mathrm{Ch}(X)$; then we can rearrange the coordinates $d_\mu$ such that $d_\nu = c_\nu^{p^e}$ for $\nu = 1, \cdots, t'$ and $d_\nu = 0$ for $\nu = t' + 1, \cdots, t$. Therefore there is a projection of the ambient space $P^t$ of $U$ onto $P^{t'}$ such that the image of $U$ is the variety $U^*$, where $U^*$ is the locus of $(\mathrm{Ch}(X'))^{(p^e)}$ over $k$ and clearly $\deg U = \deg U^*$. By Lemma 3 $\deg U^* = \deg U'$ and $\deg U' = h^r$ by Case 1, which completes the proof.

**2.** Let $V^n$ be a complete, projective variety defined over $k$.

LEMMA 4. *Let $\mathcal{L}$ be a linear system of divisors on $V$ having the following properties:*

1. *If $P$, $P'$ are any two simple points on $V$, then the linear subsystem $\mathcal{L}(P,P')$ of $\mathcal{L}$, consisting of all divisors of $\mathcal{L}$ going through $P$ and $P'$, has no base points (except $P$ and $P'$).*

2. *If $K \supset k$ is any field such that $\mathcal{L}$ has an associated function module with a basis of functions defined over $K$ and if $Y_1, \cdots, Y_n$ are $n$ independent generic members of $\mathcal{L}(P,P')$ over $K(P,P')$, then $Y_1 \cdot \cdots \cdot Y_n$ is defined and equal to $1 \cdot P + 1 \cdot P' + W$, where $W$ is a cycle not containing $P$ and $P'$.*

*Under these assumptions, if $X_1, \cdots, X_n$ are $n$ independent generic members of $\mathcal{L}(P)$ over $K(P)$, then:*

a. *$X_1 \cdot \cdots \cdot X_n$ is defined and equal to $1 \cdot P + 1 \cdot Q_1 + \cdots + 1 \cdot Q_q$,*

b. *if $i \neq j$, then $Q_i$ and $Q_j$ are independent generic points of $V$ over $K(P)$.*

*Proof.* First, we remark that if $A$ is a subvariety of $V$ algebraic over $K(P)$ and different from $V$ itself, then $X_1 \cap \cdots \cap X_n \cap A = \emptyset$. For it follows from 1. that $\mathcal{L}(P)$ has no fixed points and therefore a generic $X_1$ does not contain $A$, and it follows that the dimension of every component of $X_1 \cap A$ is smaller than the dimension of $A$. By repeating the argument we see that $X_1 \cap X_2 \cap \cdots \cap X_n \cap A = \emptyset$. Now let $Q \in X_1 \cap \cdots \cap X_n$ $(Q \neq P)$; then it follows from the above remark that $Q$ is a simple point on $V$. Next, let $Y_1, \cdots, Y_n$ be as in 2. for $\mathcal{L}(P, Q)$; we have $(X_1, \cdots, X_n) \rightarrow (Y_1, \cdots, Y_n)$ over $K(P)$. Therefore $X_1 \cdot \cdots \cdot X_n$ is defined, and $P$ and at least one other

point in $X_1 \cdot \cdots \cdot X_n$ have multiplicity 1. Put $K_1 = K(P)$; let $\phi_1, \cdots, \phi_N$ be a base for a function module of $\mathcal{L}(P)$ such that all $\phi_i$ are defined over $K_1$; moreover, we can assume that all $\phi_i$ are defined at $Q$ and that at least one $\phi_i$, say $\phi_1$, is such that $\phi_1(Q) \neq 0$ (since $Q$ is no base point for $\mathcal{L}(P)$). Let $X_i = (\sum_{\rho=1}^{N} u_{i\rho}\phi_\rho)_0$, $i = 1, \cdots, n$, with the $u_{i\rho}$ independent transcendentals over $K_1$. We have $\dim_{K_1} K_1(u, Q) = \dim_{K_1} K_1(u) = Nn$. Since

$$u_{i1} \in K_1(Q, u_{i2}, \cdots, u_{iN}) \text{ for } i = 1, \cdots, n,$$

we see that $\dim_{K_1(Q)} K_1(Q, u) \leqq (N-1)n$. Then it follows from the tower $K_1 \subset K_1(Q) \subset K_1(Q, u)$ that

$$\dim_{K_1} K_1(Q) = n \text{ and } \dim_{K_1(Q)} K_1(Q, u) = (N-1)n.$$

Then $X_1, \cdots, X_n$ are $n$ independent generic members of $\mathcal{L}(P, Q)$ over $K(P, Q)$; for, the dimension of $\mathcal{L}(P)$ being $N$ and $Q$ being not a fixed point of $\mathcal{L}(P)$, the dimension of $\mathcal{L}(P, Q)$ is $N-1$. Then a. follows from 2. applied to $\mathcal{L}(P, Q)$. Now let $Q_i$ and $Q_j$ be as in b. From what we just have seen it follows that $X_1, \cdots, X_n$ are $n$ independent members of $\mathcal{L}(P, Q_i)$ over $K(P, Q_i)$. If the locus of $Q_j$ over this field has a dimension smaller than $n$, then, since $\mathcal{L}(P, Q_i)$ has by assumption no base points (except $P$ and $Q_i$), it follows by the same argument as in the beginning of this proof that $X_1 \cap \cdots \cap X_n$ has an empty intersection with every component of the locus of $Q_j$. This being a contradiction, it follows that $Q_j$ has dimension $n$ over $K(P, Q_i)$.

LEMMA 5. *The linear system $\mathcal{L}_m$ $(m > 1)$ of hypersurface sections of degree $m$ on $V$ is ample and has the properties* 1. *and* 2. *of Lemma* 4.

*Proof.* It is well known that $\mathcal{L}_m$ is ample. Given $P$ and $P' \in V$ and an arbitrary point $Q$ in the ambient projective space $P^N$ of $V$, then there clearly exists a hypersurface of degree $m$ (if $m > 1$) through $P$ and $P'$ but not through $Q$. As to property 2., let $H_m^{(i)} = L^{(i)} + H_{m-1}^{(i)}$, $(i = 1, \cdots, n)$, where $L^{(i)}$ is a hyperplane through $P$ and $H_{m-1}^{(i)}$ is a hypersurface of degree $m-1$ through $P'$, and moreover we take generic $L^{(i)}$ and $H_{m-1}^{(i)}$ with these properties and all independent from each other (over a field $K(P, P')$ as in Lemma 4). Since in particular $L^{(i)}$ does not go through $P'$ and $H^{(i)}$ not through $P$, we have if we put $V \cdot H_m^{(i)} = X_i'$ that $X_1' \cdot \cdots \cdot X_n'$ has property 2 in Lemma 4, so certainly for generic $X_i$ we have this property.

*Definition.* A variety $V^n$ is *adaptable embedded* in projective space $P^N$ if $V \subset P^N$ and if this embedding has the following property. If $k$ is a field of definition for $V$ and $P$ a simple point on $V$ rational over $k$, and if $L^{N-n}$ is a linear variety through $P$ but otherwise generic over $k$, then:

1. $V \cdot L = P + \sum_{i=2}^{h} Q_i$ with $Q_i \neq Q_j$ for $i \neq j$ $(h = \deg V)$,

2. for every pair $(i, j)$ with $i \neq j$, the points $Q_i$ and $Q_j$ are independent generic points of $V$ over $k$.

LEMMA 6. *Let $V^n$ be a variety in projective space $P^N$. Let $V'$ be the embedding of $V$ into projective space $P^M$ by means of the hypersurface sections of degree $m$ $(m > 1)$. Then $V'$ is adaptable embedded in $P^M$.*

*Proof.* Instead of considering $V' \cdot L$ in $P^M$ we can consider in $P^N$ the intersection $V \cdot H$, where $H$ is a complete intersection of $n$ hypersurfaces of degree $m$, independent generic from each other over the field $k$ in consideration (except for going through the given point $P$). The lemma follows then from Lemma 5 and Lemma 4.

**3.** LEMMA 7. *Let $U$ be the Chow variety of an algebraic system of positive r-dimensional cycles in $P^N$, defined over a field $k$ with a generic element $X$ over $k$. Let $Z$ be a $k$-rational positive $r$-dim. cycle. Let $W$ be the locus of $\mathrm{Ch}(X + Z)$ over $k$. Then there is an everywhere biregular birational transformation between $U$ and $W$.*

*Proof.* Let $U$ be in $P^t$. Let $\sum_{\lambda=0}^{t} \xi_\lambda M_\lambda(U)$ be the Chow form of $X$ and let $F(U) = \sum \xi_\lambda^* M_\lambda(U)$ be a form with generic $(\xi_\lambda^*)$ over $k$. Suppose $\sum \alpha_\nu N_\nu(U) = G(U)$ is the Chow form of $Z$. Put $F(U) \cdot G(U) = H(U) = \sum_{\mu=0}^{s} \eta_\mu L_\mu(U)$. Then $W$ is in $P^s$. Furthermore, $\eta_\mu = \rho_\mu(\xi^*)$, where the $\rho_\mu$ are linear forms in the $\xi^*$. These forms define a projective transformation of $P^t$ into $P^s$; since it follows clearly, from the way the $\eta_\mu$ are defined, that there is no $(\xi')$ such that all $\rho_\mu(\xi')$ are zero, it follows that $P^t$ is transformed in a one-to-one manner to a subspace of $P^s$. $W$ is then clearly the projective transformation of $U$.[2]

PROPOSITION 1. *Let $V$ be a complete variety, non-singular in codimension 1 and adaptable embedded in projective space. The the Chow variety of a linear system is non-singular.*

*Proof.* Let $U^r$ be the Chow variety of a linear system $\mathcal{L}$; let $\deg V^n = h$. By Lemma 7 we can assume that $\mathcal{L}$ has no fixed components. Let $X \in \mathcal{L}$; since $\mathcal{L}$ has no fixed components, there is a point $P \in |X|$ which is not a base point for $\mathcal{L}$; we can assume that $P$ is simple on $V$. Let $k$ be a field of definition for $V$ and $U$, such that $X$ and $P$ are rational over $k$ and such that $\mathcal{L}$ has a function module with a basis of functions defined over $k$. Let $L^{N-n}$

[2] It follows from this that the assumption in Lemma 2 that $\mathcal{L}$ has no fixed components can be omitted.

be a linear variety in the ambient space $P^N$ of $V$, going through $P$ but otherwise generic over $k$ defined by a set of elements $(u_{ij})$ $(j=0,\cdots,N;$ $i=1,\cdots,n)$. Let $V\cdot L=P+\sum_{i=2}^{h}Q_i$ (by assumption on the embedding $Q_i\neq Q_j\neq P$). Let $H$ be the hyperplane in the ambient space $P^t$ of $U$ derived from the set $(u_{ij})$; $U\cap H$ consists of all divisors in $\mathcal{L}$ which contain at least one point of $V\cap L$. Denote by $\mathcal{L}_P$, resp. $\mathcal{L}_{Q_i}$ $(i=2,\cdots,h)$, the linear subsystems of $\mathcal{L}$ consisting of all divisors through $P$, resp. $Q_i$. $\mathcal{L}_P$ is a proper subsystem of $\mathcal{L}$ (since $P$ is not a base point) and also the $\mathcal{L}_{Q_i}$ are proper; in fact, $X\notin\mathcal{L}_{Q_i}$ since $Q_i$ is generic on $V$ over $k$ and $X$ is rational over $k$. Moreover, for $i\neq j$ we have $\mathcal{L}_{Q_i}\neq\mathcal{L}_{Q_j}$ since $Q_i$ and $Q_j$ are independent generic points of $V$ over $k$ (since $V$ is adaptable embedded). Let $U_P$, $U_i$ $(i=2,\cdots,h)$ be the Chow varieties of $\mathcal{L}_P$ and $\mathcal{L}_{Q_i}$ respectively. Then the above stated properties can be translated as follows

$$U\cap H=U_P\cup U_2\cup\cdots\cup U_h,\quad \mathrm{Ch}(X)\notin U_i\quad (i=2,\cdots,h)$$

and $U_i\neq U_j$ for $i\neq j$. By Lemma 2 $\deg U=h^r$; by Lemma 1 $\deg U_P\geqq h^{r-1}$ and $\deg U_i\geqq h^{r-1}$ $(i=2,\cdots,h)$. Therefore we most have

$$U\cdot H=1\cdot U_P+\sum_{i=2}^{h}1\cdot U_i.$$

By the criterion of multiplicity 1 [8; VI, Th. 6] the proposition is proved if $\dim U=1$, and if $\dim U>1$, it suffices to prove (since $\mathrm{Ch}(X)\notin U_i$) that $\mathrm{Ch}(X)$ is simple on $U_P$. Therefore proceeding by induction on $\dim U$ the proof is complete.

PROPOSITION 2.[3] *Let $V^n$ be a complete, non-singular projective variety. Let $U$ be the Chow variety of a maximal, total, regular family $\mathcal{U}$ of positive divisors on $V$. Let $X\in\mathcal{U}$; then the Chow variety of the complete linear system $\mathcal{L}(X)$, determined by $X$, is a simple subvariety of $U$.*

*Proof.* Let $\mathrm{Ch}(X_0)$ be a simple point of $U$ and let k be a common field of definition for $V$, $U$ and the Pic.$(V)$ over which $X$ and $X_0$ are rational. Consider the rational mapping $h: U\to\mathrm{Pic.}(V)$ defined by $h(\mathrm{Ch}(X^*))=\mathrm{Class}(X^*-X_0)$, where $X^*$ is a generic element of $\mathcal{U}$ over $k$; $h$ is defined over $k$. Let $\Gamma_h$ be the graph of $h$ on $U\times\mathrm{Pic.}(V)$. Let $\mathrm{Ch}(X^*)$ be a generic point of $U$ over $k$, put $\eta^*=\mathrm{Cl.}(X^*-X_0)$ and $\eta=\mathrm{Cl.}(X-X_0)$. If we denote by $\mathcal{L}(X^*)$, resp. $\mathcal{L}(X)$, also the Chow varieties of the complete linear systems determined by $X^*$, resp. $X$, then we have by [6, I, Prop. 10 and Cor.] $\Gamma_h\cdot(P^t\times\eta^*)=\mathcal{L}(X^*)\times\eta^*$ and $\Gamma_h\cap(P^t\times\eta)=\mathcal{L}(X)\times\eta$. Since $\mathcal{U}$ is a

[3] The writer owes this proposition to T. Matsusaka; it is of special interest for it follows from this proposition that the Picard variety can be constructed in precisely the same way as the Jacobian variety is constructed by Chow in [3]. (See the remark following this proposition.)

regular family, all the linear systems have the same dimension, and then it follows by Lemma 2 (and footnote 2) that the corresponding Chow varieties have the same degree. Therefore we must have $\Gamma_h \cdot (P^t \times \eta) = \mathcal{L}(X) \times \eta$ (and not a multiple of that cycle since this contradicts the fact that $\Gamma_h \cdot (P^t \times \eta)$ is the specialization of $\Gamma_h \cdot (P^t \times \eta^*)$ over the specialization $\eta^* \to \eta$ with respect to $k$). Hence by the criterion of multiplicity 1 [8, VI, Th. 6] we see that $\mathcal{L}(X) \times \eta$ is a simple subvariety of $\Gamma_h$. It suffices therefore to show that the mapping $h$ is regular (in the sense of [8]) at the subvariety $\mathcal{L}(X)$ of $U$.

Let $C$ be a generic 1-section of $V$ over $k$, $J$ the Jacobian of $C$ and $\phi\colon C \to J$ the canonical mapping. Let $B$ be the abelian subvariety of $J$ generated by the points $S\phi((X^* - X_0) \cdot C)$. Then by [4] [4] $B$ is a model for the Pic.$(V)$ and the mapping $h'\colon U \to B$ defined by $h'(\mathrm{Ch}(X^*)) = S\phi((X^* - X_0) \cdot C)$ is the canonical mapping; hence we can take Pic.$(V) = B$ and $h = h'$. If $K \supset k$ is such that $C$, $J$, $\phi$ are defined over $K$, then we must show that $h'$ is regular at $\mathrm{Ch}(X')$, where $\mathrm{Ch}(X')$ is a generic point of $\mathcal{L}(X)$ over $K$. In view of its application in the next theorem we state the next lemma.

LEMMA 8. *If $C$ is a generic* 1*-section of $V$ over $k(\mathrm{Ch}(X'))$ and if $J$, $\phi$, $B$ and $h'$ are introduced as above, then $h'$ is regular at* $\mathrm{Ch}(X')$.

*Proof.* Instead of considering $h'$ we can consider $f\colon U \to J$ defined by $f(\mathrm{Ch}(X^*)) = S\phi(X^* \cdot C)$ since $h'$ and $f$ differ only by a constant on $J$. Let $\deg(X^* \cdot C)$ be $d$. Then we have the following commutative diagram:

$$\begin{tikzcd} U \arrow[rr, "f"] \arrow[dr, "g"'] & & J \\ & C^{(d)} \arrow[ur, "\psi"'] & \end{tikzcd}$$

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[4] Since [4] is still unpublished, we indicate how the proof here can also be obtained from Chow's results. It is irrelevant for our considerations that Pic.$(V)$ is embedded in one Jacobian, an embedding into a product of Jacobians is sufficient. By the so-called regularity theorem [Lang, Abelian Varieties, VIII, Th. 9] there is such an embedding. However, we must also have a connection between the natural mapping $X \to \mathrm{Cl.}(X - X_0)$ and the intersection of $X$ with the generic 1-sections. In fact, we must have commutativity in the following diagram (where we can restrict to one Jacobian),

$$\begin{tikzcd} U \arrow[rr, "\psi"] \arrow[dr, "h"'] & & J_u \\ & \mathrm{Pic.}(V) \arrow[ur, "\lambda_u"'] & \end{tikzcd}$$

where $h$ is the natural mapping $X \to \mathrm{Cl.}(X - X_0)$, $\lambda_u$ is the canonical mapping of the $k(u)/k$-trace Pic.$(V)$ to $J_u$ and $\psi(Ch(X)) = S\phi((X - X_0) \cdot C_u)$. This commutativity follows essentially from Th. 12 and Th. 4 in Lang's book, Chap. VIII.

where $C^{(d)}$ is the Chow variety of positive divisors of degree $d$ on $C$, $g$ is the mapping $g(\mathrm{Ch}(X^*)) = \mathrm{Ch}(X^*\cdot C)$ and $\psi$ is the mapping $\psi(\sum_{j=1}^{d} P_j) = S\phi(P_j)$, where $P_j$ are points on $C$. Since $C^{(d)}$ is non-singular,[5] it suffices to prove that $g$ is regular at $\mathrm{Ch}(X')$. Let the Chow form of $X'$ be $\sum_\lambda \xi_\lambda M_\lambda(U)$ where the $M_\lambda(U)$ are monomials in the letters $U_{ij}$ $(j=0,\cdots,N;\ i=1,\cdots,n)$. Let $C$ be the intersection of $V$ with a linear space $L$ defined by $\sum_{j=0}^{N} v_{ij}Z_j = 0$ $(i=1,\cdots,n-1)$ with $v_{ij}$ independent transcendentals over $k(\mathrm{Ch}(X'))$. Putting $L\cdot X' = \sum_{\alpha=1}^{d} P_\alpha$ with $P_\alpha = (p_{\alpha 0},\cdots,p_{\alpha N})$ we have by the definitions of the Chow forms of $X'$ and $X'\cdot L$ the relation

$$\sum \xi_\lambda M_\lambda(v,\cdots U_{nj}\cdots) = \rho(v)\prod_{\alpha=1}^{d}(\sum_{j=0}^{N} U_{nj}p_{\alpha j}) = \rho(v)\sum_\mu \eta_\mu N_\mu(U),$$

where $\rho(v)$ is a rational function in the $v$'s and $\sum_\mu \eta_\mu N_\mu(U)$ is the Chow form of $X'\cdot L$ (in the letters $U_{nj}$). Therefore the $\eta_\mu$ are polynomials in the $\xi_\lambda$ (with coefficients in $k(v)$) and since, of course, not all these polynomials are zero, the mapping $g$ is regular at $\mathrm{Ch}(X')$.

*Remark.* It follows from Proposition 2 that the variety $W$ of [6, II, page 59] is itself a model for the Pic.$(V)$. It follows from the properties stated there that is suffices, in order to show this, that $W$ is non-singular. $W$ is the Chow variety of the family of linear systems $\mathcal{L}(X)$ on $U$. Since $\mathfrak{U}$ is regular, every linear system has the same dimension and therefore by Lemma 2 the same degree. Therefore we have an involutional system in the sense of [2]. By a theorem of Chow [2, p. 258] it suffices to show that each $\mathcal{L}(X)$ is a simple subvariety of $U$, but this is precisely the assertion of Proposition 2.[6]

THEOREM. *Let $V$ be a complete, non-singular variety, adaptable embedded in projective space. Then the Chow variety of a maximal, total, regular family of positive divisors on $V$ is non-singular.*

*Proof.* Let $U$ be the Chow variety under consideration; let $\mathrm{Ch}(X) \in U$. Let $\mathrm{Ch}(X_0)$ be a simple point on $U$. Consider as in the proof of Proposition 2

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[5] There is an oversight on page 472 of [3] in the proof of the non-singularity of $C^{(d)}$ (in case the divisor has multiple components). However, by using the same arguments as in the proof of Proposition 1 this can be corrected; the essential point being that we know the degree of the Chow variety $C^{(d)}$ in terms of the degree of $C$.

[6] The method of Matsusaka for constructing the Picard variety can also be used to construct the so-called "generalized Picard varieties" in the sense of Tate (see L. Lang, Abelian Varieties, page 176). We hope to return to this question on some future occasion.

the mapping $h: U \to \mathrm{Pic.}(V)$ defined by $h(\mathrm{Ch}(X^*)) = \mathrm{Cl.}(X^* - X_0)$, where $\mathrm{Ch}(X^*)$ is a generic point of $U$ (over a field $k$ of definition for $V$ and $U$ over which $X$ and $X_0$ are rational). Let $\eta = \mathrm{Cl.}(X - X_0)$; then, if $\Gamma_h$ is the graph of $h$, we have seen that $\Gamma_h \cdot (P^t \times \eta) = \mathfrak{L}(X) \times \eta$ ($P^t$ is, as usual, the ambient space of $U$, $\mathfrak{L}(X)$ is the Chow variety of the linear system determined by $X$). First, we want to show that $\mathrm{Ch}(X) \times \eta$ is simple on $\Gamma_h$. By [8, VI, Th. 6] it suffices to show that $\mathrm{Ch}(X) \times \eta$ is simple on $\mathfrak{L}(X) \times \eta$. This follows from the fact that $\mathfrak{L}(X)$ is non-singular by Proposition 1. Therefore it suffices to show that $h$ is regular at $\mathrm{Ch}(X)$. Introducing a generic 1-section $C$ of $V$ over $k$ and $J$ and $B$ and $h'$ as above we can take by [4] Pic. $V = B$ and $h = h'$. Then $h'$ is regular at $\mathrm{Ch}(X)$ by Lemma 8.

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# ON THE ALGEBRA OF REPRESENTATIVE FUNCTIONS OF AN ANALYTIC GROUP.*

By G. Hochschild and G. D. Mostow.

1. **Introduction.** Let $G$ be a real or complex analytic group. If $\rho$ is an analytic finite dimensional representation of $G$, and if $t$ is a linear functional on the algebra of all linear endomorphisms of the representation space of $\rho$, then the composite $t \circ \rho$ is called a representative function on $G$ associated with $\rho$. Throughout, $R$, or $R(G)$, if the group is to be exhibited, denotes the algebra of all complex valued representative functions on $G$. In the analysis of $R$, a special role is played by the group $\mathrm{Hom}(G, C)$ of all analytic homomorphisms of $G$ into the additive group $C$ of the complex numbers. By composition with the exponential map of $C$ into the multiplicative group $C^*$ of the non-zero complex numbers, we obtain the subgroup $Q = \exp(\mathrm{Hom}(G, C))$ of the group $\mathrm{Hom}(G, C^*)$ of all analytic homomorphisms of $G$ into $C^*$. It is a fundamental feature of the generalized Tannaka Theorem (see [2], [3], [4]) that the departure of the representation theory of $G$ from that of an algebraic linear group depends entirely on the non-triviality of $Q$. It is for this reason that $Q$ is singled out in a natural way in the structure of $R$.

A subalgebra $B$ of $R$ will be called a *basic subalgebra* if it satisfies the following conditions: (1) $B$ contains the constants, (2) $B[Q] = R$, (3) the elements of $Q$ are free over $B$, (4) in the real case, $B$ is stable under the complex conjugation of $R$. It is known from [3] and [4] that *there always exists a finitely generated basic subalgebra.* However, this result is inadequate in as much as it ignores the $G$-module structure of $R$. Actually, we shall show in Section 3 that there always exists a basic subalgebra that is stable under the left $G$-translations and, in fact, has the additional stability property that its 'semisimple part' is stable under both the left and the right $G$-translations. Although it is easy to see (Section 3) that all basic subalgebras are isomorphic as algebras, they may differ radically in their behaviour under the action of $G$. On the other hand, we shall show in Section 4 that the *normal basic subalgebras,* i. e., the basic algebras that are left stable and whose semisimple part is two-sidedly stable, can be classified into orbits under the right $G$-translations, which correspond in a natural 1-1 fashion to the con-

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jugacy classes of the decompositions of $G$ into semidirect products of the type described in Section 2.

In Section 6, we analyze the group of the proper automorphisms of $R$ (i. e., the automorphisms leaving the constants fixed and commuting with the right translations) by means of a stable basic subalgebra. In particular, we show that this group is a semidirect product of a subgroup naturally isomorphic with $\mathrm{Hom}(Q, C^*)$ by the normal subgroup of the left translations (complexified, in the real case). Section 7 is a direct application of the existence of stable basic subalgebras and concerns the representations of $G$ as *closed* subgroups of full linear groups.

In Section 8, we show that the *two-sidedly stable* basic subalgebras (which do not always exist) correspond in a 1-1 fashion to the rational equivalence classes of faithful representations of the *complex* analytic group $G$ as an algebraic linear group. In particular, it follows from this and the correspondence between normal basic subalgebras and decompositions of $G$ that any two such 'algebraic structures' of $G$ are conjugate by an analytic automorphism of $G$, whence we obtain a description of the set of all algebraic structures on $G$ which exhibits this set as an affine space in a natural way. On the level of the decomposition theory of $G$, the conjugacy result is due to B. Kostant (Theorem 8.4).

The statements of the results obtained here and the main features of their proofs are intelligible without reference to our previous papers ([2], [3], [4]) on this topic. Nevertheless, we lean heavily on the notions and techniques of these papers, and we have not covered all the details of the proofs by explicit references.

**2. Nuclei and decompositions.** For later use, we review some known results concerning decompositions of analytic groups into semidirect products.

We shall say that a (real or complex) Lie group $G$ is *reductive* if $G$ has a faithful finite dimensional analytic representation and if every finite dimensional analytic representation of $G$ is semisimple. By a *nucleus* of a Lie group $G$ we shall mean a closed, normal, solvable and simply connected analytic subgroup $K$ of $G$ such that $G/K$ is reductive. Let $N$ denote the radical of the commutator subgroup $G'$ of $G$. Then every finite dimensional analytic representation of $G$ is unipotent on $N$. Now suppose that $G$ has a nucleus $K$. Then $G$ has a finite dimensional semisimple analytic representation whose kernel is exactly $K$. Since the restriction of a semisimple representation to a normal subgroup is still semisimple, we conclude that $N \subset K$. Thus *$N$ is contained in every nucleus of $G$.*

*If $G$ is an analytic group that has a faithful finite dimensional analytic representation then $G$ has a nucleus. Moreover, if $K$ is any nucleus of $G$ then $G$ is a semidirect product $H \cdot K$, where $H$ is a closed analytic subgroup of $G$ and, of course, is reductive.*

In the complex case, the existence of a nucleus $K$ for an analytic group $G$ implies the existence of a semidirect product decomposition $G = H \cdot K$, even when it is not assumed that $G$ is faithfully representable, and the existence of a faithful representation is then a consequence of the existence of a nucleus [4, Th. 3.6]. In the real case, the proof of the second assertion above is contained in the proof of [2, Th. 9.1]; one merely has to observe that any given nucleus of $G$ may take the place of the group $K$ used in that proof. The first of our two assertions above, in the real case, is part of [2, Th. 9.1]; in the complex case, the existence of a nucleus is part of [4, Th. 4.2].

In our later applications of nuclei, we shall use the fact that the Lie algebra of $K$ can be written as a sum of the Lie algebra of $N$ and another nilpotent Lie algebra that lies in the centralizer of the Lie algebra of $H$. What we shall need is contained in the following lemma.

LEMMA 2.1. *Let $\mathfrak{G}$ be a Lie algebra that is a semidirect sum $\mathfrak{H} + \mathfrak{K}$, where $\mathfrak{K}$ is a solvable ideal and $\mathfrak{H}$ is a complementary subalgebra that is reductive in $\mathfrak{G}$. Let $\mathfrak{N} = [\mathfrak{G}, \mathfrak{K}]$. Then there is a nilpotent subalgebra $\mathfrak{P}$ of $\mathfrak{K}$ such that $[\mathfrak{H}, \mathfrak{P}] = (0)$ and $\mathfrak{K} = \mathfrak{P} + \mathfrak{N}$ (not necessarily semidirect).*

*Proof.* Let $\mathfrak{L}$ denote the centralizer of $\mathfrak{H}$ in $\mathfrak{K}$. Since $\mathfrak{K}$ is semisimple as an $\mathfrak{H}$-module (under the adjoint representation), it is clear that $\mathfrak{K} = \mathfrak{L} + \mathfrak{N}$. For $x \in \mathfrak{L}$, denote by $\mathfrak{L}^x$ the subspace of all elements of $\mathfrak{L}$ that are annihilated by some power of the inner derivation effected by $x$. If we choose $x$ so that $\mathfrak{L}^x$ is of the smallest possible dimension then $\mathfrak{L}^x$ (is a Cartan subalgebra of $\mathfrak{L}$ and, in particular,) is a nilpotent subalgebra $\mathfrak{P}$ of $\mathfrak{L}$. By Fitting's Lemma, we have $\mathfrak{L} = \mathfrak{P} + \mathfrak{S}$, where $\mathfrak{S}$ is a subspace such that $[x, \mathfrak{S}] = \mathfrak{S}$. Hence $\mathfrak{S} \subset \mathfrak{N}$, and we conclude that $\mathfrak{K} = \mathfrak{P} + \mathfrak{N}$, completing the proof.

**3. Basic subalgebras.** We begin with two elementary facts concerning basic subalgebras that are important for our purpose.

PROPOSITION 3.1. *Let $U$ and $V$ be any two basic subalgebras of $R$. Then there exists a unitary $C$-algebra isomorphism of $U$ onto $V$.*

*Proof.* For every $f \in R$, write $f = \sum_{q \in Q} v_q(f) q$, with $v_q(f) \in V$. Similarly, define the maps $u_q \colon R \to U$, for each $q \in Q$. Now define the map $\phi \colon U \to V$ by $\phi(f) = \sum_{q \in Q} v_q(f)$. Then $\phi$ is evidently a unitary $C$-algebra homomorphism

of $U$ into $V$. Interchanging the roles of $U$ and $V$, we obtain a unitary $C$-algebra homomorphism $\psi\colon V \to U$. Let $f \in U$. Then we have

$$\begin{aligned}\psi(\phi(f)) &= \Sigma_q u_q(\phi(f)) = \Sigma_{q,q'} u_q(v_{q'}(f))\\ &= \Sigma_{q,q'} u_{qq'}(v_{q'}(f)q')\\ &= \Sigma_q u_q(\Sigma_{q'} v_{q'}(f)q')\\ &= \Sigma_q u_q(f) = f.\end{aligned}$$

Thus $\psi \circ \phi$ is the identity map on $U$, and similarly $\phi \circ \psi$ is the identity map on $V$. Hence $\phi$ is a $C$-algebra isomorphism of $U$ onto $V$.

PROPOSITION 3.2. *Let $U$ be a basic subalgebra of $R$ that is stable under the left (or right) translations with the elements of $G$. Then* $\mathrm{Hom}(G, C) \subset U$.

*Proof.* Let $h \in \mathrm{Hom}(G, C)$ and write $h = \Sigma_{q \in Q} u_q(h)q$, with $u_q(h) \in U$. Translating on the left with an element $x \in G$, we obtain from this

$$h(x) + h = \Sigma_{q \in Q}(x \cdot u_q(h))q(x)q.$$

Comparing coefficients, we get

$$u_q(h) = (x \cdot u_q(h))q(x), \text{ for every } q \neq 1.$$

Now evaluate at the identity element 1 of $G$. This yields

$$u_q(h)(1) = u_q(h)(x)q(x).$$

Thus, for every $q \neq 1$, $u_q(h)q$ is a constant. We conclude that $u_q(h) = 0$, for every $q \neq 1$, whence $h = u_1(h) \in U$, q. e. d.

In studying the algebra $R = R(G)$ of the complex valued representative functions on the (real or complex) analytic group $G$, we may assume (in virtue of [2, Th. 7.1] and [4, pp. 89-90]) without loss of generality that $G$ has a faithful finite dimensional analytic representation. This assumption will be in force from now on. Let $G = H \cdot K$ be a semidirect decomposition as discussed in Section 2. Let $\mathfrak{G}$, $\mathfrak{H}$, $\mathfrak{K}$ be the Lie algebras of $G$, $H$, $K$, respectively. Let $N$ be the radical of the commutator subgroup $G'$ of $G$. Then $N \subset K$, and the Lie algebra $\mathfrak{N}$ of $N$ coincides with $[\mathfrak{G}, \mathfrak{K}]$. Let $x_1, \cdots, x_m$ be a basis for $\mathfrak{N}$. Let $x_{m+1}, \cdots, x_n$ be elements of the nilpotent algebra $\mathfrak{P}$ of Lemma 2.1 such that $x_1, \cdots, x_n$ is a basis for $\mathfrak{K}$. Now every element of $G$ can be written uniquely in the form

$$h \exp(c_n x_n) \cdots \exp(c_{m+1}x_{m+1})\exp(\sum_{i=1}^{m} c_i x_i),$$

where $h \in H$ and the $c_j$ are real or complex numbers (note that $N$ is simply

connected and nilpotent). We define functions $u_1, \cdots, u_n$ on $G$ such that, for each $j$, the value of $u_j$ at the element of $G$ written above is $c_j$. Let $S$ be the algebra of functions that are generated by the constants and $u_1, \cdots, u_n$. Let $R^K$ denote the subalgebra of $R$ that consists of the elements left fixed by the translations with the elements of $K$. The analysis of $R$ made in [3, Section 4], which would be the same in the complex case as it was in the real case, has shown that $R^K S$ is a basic subalgebra of $R$ (the notation of [3] is such that our present $R^K$ is there denoted $R^H(G)$; the notation we adopt here is based on the principle that if $W$ is any left module for a group $A$ then $W^A$ stands for the submodule of the $A$-fixed elements of $W$).

We claim that $S$ is stable under the left translations with the elements of $G$. For $j > m$, we have $u_j \in \operatorname{Hom}(G, C)$, so that $g \cdot u_j = u_j(g) + u_j \in S$, for every $g \in G$. Now suppose that $j \leqq m$. Let $v \in H$. Then $v$ commutes with every $\exp(c_k x_k)$ with $k > m$. Hence, if $t \in \mathfrak{N}$ and $v^*$ denotes the automorphism of $\mathfrak{N}$ that corresponds to $v^{-1}$ under the adjoint representation,

$$
\begin{aligned}
& h \exp(c_n x_n) \cdots \exp(c_{m+1} x_{m+1}) \exp(t) v \\
& \qquad = hv \exp(c_n x_n) \cdots \exp(c_{m+1} x_{m+1}) v^{-1} \exp(t) v \\
& \qquad = hv \exp(c_n x_n) \cdots \exp(c_{m+1} x_{m+1}) \exp(v^*(t)).
\end{aligned}
$$

We see at once from this that $v \cdot u_j$ is a linear combination of $u_1, \cdots, u_m$. Now let $s \in \mathfrak{N}$, and let us consider the translate $\exp(s) \cdot u_j$. The nilpotency of $\mathfrak{N}$ implies that, if $t \in \mathfrak{N}$, we have

$$\exp(t)\exp(s) = \exp(f(s, t)),$$

where $f$ is a polynomial map of $(\mathfrak{N}, \mathfrak{N})$ into $\mathfrak{N}$. Hence it is clear that $\exp(s) \cdot u_j$ is a polynomial in $u_1, \cdots, u_m$.

There remains to consider the translates $\exp(d_k x_k) \cdot u_j$, where $k > m$ and $d_k$ is an arbitrary real or complex number. Since $\mathfrak{P}$ is nilpotent, we have

$$
\begin{aligned}
& \exp(c_n x_n) \cdots \exp(c_{m+1} x_{m+1}) \exp(d_k x_k) \\
& \qquad = \exp(c_n x_n) \cdots \exp((c_k + d_k) x_k) \cdots \exp(c_{m+1} x_{m+1}) \exp(s),
\end{aligned}
$$

where $s$ is a linear combination of basis elements of $[\mathfrak{P}, \mathfrak{P}]$ whose coefficients are polynomials in $c_{m+1}, \cdots, c_n$ and $d_k$. Writing $t$ for $\sum_{i=1}^{m} c_i x_i$, we have

$$
\begin{aligned}
& h \exp(c_n x_n) \cdots \exp(c_{m+1} x_{m+1}) \exp(t) exp(d_k x_k) \\
& = h \exp(c_n x_n) \cdots \exp((c_k + d_k) x_k) \cdots \exp(c_{m+1} x_{m+1}) \exp(s) \exp(\exp(d_k x_k)^*(t)).
\end{aligned}
$$

Since $s \in \mathfrak{N}$, the product of the last two factors is the exponential of $f(s, \exp(d_k x_k)^*(t))$, which is a linear combination of $x_1, \cdots, x_m$ whose coefficients are polynomials in $c_1, \cdots, c_n$ (the dependence on $d_k$ being ignored). Hence $\exp(d_k x_k) \cdot u_j$ is a polynomial in $u_1, \cdots, u_n$.

Thus we have shown that $S$ is stable under the left translations. Evidently, $R^K$ is stable under the left and the right translations. Moreover, $R^K$ is canonically isomorphic with the algebra of all representative functions on the reductive analytic group $G/K$. By [2, Th. 9.2] (for the real case) and [4, Th. 5.2] (for the complex case), $R^K$ is therefore finitely generated as a $C$-algebra. Hence $R^K S$ is finitely generated as a $C$-algebra. Put $B = R^K S$.

It is easy to see (cf. [3, Section 4]) that the canonical map $R^K \otimes_C S \rightarrow R^K S$ is an isomorphism. We recall that a representative function $f$ on $G$ is called semisimple if the representation of $G$ by left translations on the space spanned by the translates of $f$ is semisimple. The semisimple elements of $B$ constitute a subalgebra $B_s$ of $B$ that is stable under the left translations. We claim that $B_s = R^K$. Evidently, $R^K \subset B_s$. Conversely, let $f \in B_s$. We can write $f = \sum_{i=1}^{r} p_i s_i$, where $p_1, \cdots, p_r$ are $C$-linearly independent elements of $R^K$ and the $s_i \in S$. Then we have, for every $x \in K$, $x \cdot f = \sum_{i=1}^{r} p_i(x \cdot s_i)$. Since, for given $p_1, \cdots, p_r$, this representation of $x \cdot f$ is unique, it follows that, for each $i$, the $K$-module spanned by the left $K$-translates of $s_i$ is a $K$-homomorphic image of the $K$-module spanned by the left $K$-translates of $f$. Since $K$ is normal in $G$, the $K$-module generated by $f$ is semisimple. Hence the $K$-module generated by $s_i$ is semisimple, for each $i$. On the other hand, $N$ is unipotent on every finite dimensional $N$-submodule of $R$. Hence $s_i$ must be left fixed by the left $N$-translations, and it follows that the restriction of $s_i$ to $K$ may be regarded as a representative function on the vector group $K/N$ and, as such, is a semisimple representative function on $K/N$. Hence the restriction of $s_i$ to $K$ is a $C$-linear combination of elements of $\exp(\mathrm{Hom}(K/N, C))$, i.e., it coincides with the restriction to $K$ of a $C$-linear combination of elements of $Q = \exp(\mathrm{Hom}(G, C))$. However, it is clear from [3, Section 4] that the restriction homomorphism $R(G) \rightarrow R(K)$ is a monomorphism on $S[Q]$. Thus we conclude that $s_i \in C[Q]$. Since the elements of $Q$ are free over $S$, this implies that $s_i$ is a constant, whence $f \in R^K$.

It is convenient to introduce the following definition: *a normal basic subalgebra of $R$ is a basic subalgebra $B$ such that $B$ is stable under the left translations and $B_s$ is stable under both the left and the right translations.*

The algebra $B$ constructed above has been shown to be a normal basic

subalgebra. Since $B$ is finitely generated as a $C$-algebra, it is clear from Propostition 3.1 that *every basic subalgebra of $R$ is finitely generated as a $C$-algebra.* Next, we observe that the kernel of the representation of $G$ by left translations on $B_s$ is the nucleus $K$ of $G$. Indeed, since $B_s = R^K$, this kernel contains $K$. Since $G = H \cdot K$, and since the representation of $H$ by left translations on $R^K$ must be faithful (because $H$ has a faithful finite dimensional representation and $R^K$ is canonically isomorphic with the algebra of all representative functions on $H$), it follows that the kernel must coincide with $K$.

Now let $D$ be any basic subalgebra of $R$ that is stable under the left translations. We claim that $R_s = D_s[Q]$. Evidently, $D_s[Q] \subset R_s$. Conversely, let $f \in R_s$ and write $f = \sum_{q \in Q} d_q(f) q$, with $d_q(f) \in D$. Then we have, for every $x \in G$, $x \cdot f = \sum_{q \in Q} (x \cdot d_q(f)) q(x) q$, and $x \cdot d_q(f) \in D$. Denote by $G \cdot f$ the space spanned by the left translates of $f$, etc. Then $(G \cdot d_q(f)) q$ is a $G$-homomorphic image of $G \cdot f$, for each $q$, and hence is semisimple. Evidently, the $G$-submodules of $(G \cdot d_q(f)) q$ are the $G$-modules $Vq$, where $V$ ranges over the $G$-submodules of $G \cdot d_q(f)$. Hence we conclude that each $d_q(f)$ is semisimple, i. e., that $d_q(f) \in D_s$. Hence $f \in D_s[Q]$, and our claim is proved.

Let $\phi$ denote the coefficient sum isomorphism of $B$ onto $D$, as in Proposition 3.1. It follows at once from what we have just seen that $\phi(B_s) \subset D_s$. Similarly, $\phi^{-1}(D_s) \subset B_s$. Hence $\phi$ maps $R^K = B_s$ isomorphically onto $D_s$. For $x \in G$, define the map $\phi_x \colon R^K \to C$ by $\phi_x(f) = \phi(f)(x)$. Then $\phi_x$ is an algebra homomorphism leaving the constants fixed. Furthermore, in the real case, $\phi_x$ evidently commutes with the complex conjugation. By [2, Prop. 2.5], $\phi_x$ defines a unique proper automorphism $\zeta$ of $R^K$ such that $\zeta(f)(1) = \phi_x(f)$. In the real case, $\zeta$ commutes with the complex conjugation. Moreover, it is clear that the map $x \to \zeta$ is continuous, so that, in the real case, $\zeta$ belongs to the connected component of the identity in the group of the real proper automorphisms of $R^K$. Now $R^K$ may be regarded as the algebra of all representative functions on the reductive analytic group $H$. Hence it follows from [2, Th. 1.1.1] (for the real case) and [4, Th. 5.2] (for the complex case) that $\zeta$ is the left translation by an element $x_1 \in H$. Thus we have $\phi(f)(x) = f(x_1)$, for every $f \in R^K$.

Now let $d \in D_s$ and write $d = \sum_{q \in Q} f_q(d) q$, with $f_q \in B_s = R^K$. Then $d(x_1) = (\sum_{q \in Q} f_q(d))(x_1)$, because $q(x_1) = 1$, for each $q$, since $x_1 \in H$. Thus $d(x_1) = \phi(\sum_{q \in Q} f_q(d))(x) = d(x)$, by the proof of Proposition 3.1.

Now suppose that $D$ is a *normal* basic subalgebra. Then, for every $d \in D_s$ and every $y \in G$, $d \cdot y \in D_s$, and hence $(d \cdot y)(x_1) = (d \cdot y)(x)$. Hence we have $x_1 \cdot d = x \cdot d$, for every $d \in D_s$. Now let $L$ denote the kernel of the

representation of $G$ by left translations on $D_s$. Then our last result shows that $G = HL$. Moreover, the isomorphism $\phi\colon R^K \to D_s$ is evidently an $H$-module isomorphism. Since the representation of $H$ by left translations on $R^K$ is faithful, it follows that the representation of $H$ by left translations on $D_s$ is also faithful. Hence $H \cap L = (1)$, and $G$ is the semidirect product $H \cdot L$. Clearly, $N \subset L$, and $L/N$ is isomorphic with $G/(HN)$ and hence with $K/N$. Hence $L$ is solvable. Moreover, since $H \cdot L = H \cdot K$, it follows that $L$ is homeomorphic with $K$, and thus is simply connected. Hence $L$ is a nucleus of $G$. Now construct a normal basic subalgebra $E$ of $R$ as above, but using the nucleus $L$. Then $E_s = R^L$, and $D_s[Q] = R_s = E_s[Q]$. Since $D_s \subset E_s$ and since the elements of $Q$ are free over $E_s$, this implies that $D_s = E_s = R^L$.

We may now summarize our results as follows.

THEOREM 3.1. *Let $G$ be an analytic group having a faithful finite dimensional analytic representation. Let $R$ be the algebra of the representative functions on $G$, and let $K$ be a nucleus of $G$. Then there exists a normal basic subalgebra $B$ of $R$ such that $B_s = R^K$ and $K$ is the kernel of the representation of $G$ by left translations on $B_s$. If $D$ is any normal basic subalgebra of $R$ then the kernel of the representation of $G$ by left translations on $D_s$ is a nucleus $L$ of $G$, and $D_s = R^L$.*

We observe that *if $G$ is solvable then every basic subalgebra of $R$ that is stable under the left translations is a normal basic subalgebra.* Indeed, if $G$ is solvable we have $G' = N$, so that $G/N$ is abelian. Now if $B$ is a left stable basic subalgebra of $R$ then the elements of $B_s$ are left fixed by the left translations with the elements of $N$. It follows that, for every $f \in B_s$ and every $x \in G$, $f \cdot x = x \cdot f$, which proves our assertion.

In general, this last result does not hold, as is shown by the following example. Let $H$ denote the group of all 2 by 2 complex matrices with determinant 1, and let $G = H \times C$. Let $\alpha$, $\beta$, $\gamma$, $\delta$ be the functions on $G$ that associate with each element of $G$ the entries of the matrix component of that element, so that $\alpha\delta - \beta\gamma = 1$. Let $\rho$ be the projection of $G$ onto $C$ with kernel $H$, and put $q = \exp(\rho)$. Then it is easily seen that a left stable basic subalgebra of $R$ is given by

$$B = C[\alpha q, \beta q, \gamma q^{-1}, \delta q^{-1}, \rho]$$

We have

$$B_s = C[\alpha q, \beta q, \gamma q^{-1}, \delta q^{-1}],$$

and one verifies directly that $B_s$ is *not* stable under the *right* translations.

Moreover, the kernel of the representation of $G$ by left translations on $B_s$ is discrete, in this case, and thus is certainly not a nucleus of $G$.

## 4. Relations among normal basic subalgebras.

LEMMA 4.1. *Let $B$ be any normal basic subalgebra of $R$, and let $K$ be the kernel of the representation of $G$ by left translations on $B_s$. Then $B^N = R^K[\mathrm{Hom}(G, C)]$.*

*Proof.* From a semidirect product decomposition $G = H \cdot K$, we see that $G/N$ is the direct product of the reductive group $(HN)/N$ by the vector group $K/N$. Hence $R(G/N)$ may be identified with the tensor product of $R(G/N)^{K/N}$ and $R(G/N)^{(HN)/N}$. Now $R(G/N)^{(HN)/N}$ is canonically isomorphic with $R(K/N) = C[\mathrm{Hom}(K/N, C), \exp(\mathrm{Hom}(K/N, C)]$. Hence we have

$$R(G/N)^{(HN)/N} = C[\mathrm{Hom}(G/N, C), \exp(\mathrm{Hom}(G/N, C)].$$

Now if we apply the canonical isomorphism of $R(G/N)$ onto $R(G)^N$ we find

$$R^N = R^K \otimes_C C[\mathrm{Hom}(G, C), Q] = R^K[\mathrm{Hom}(G, C), Q].$$

Thus we have

$$R^K[\mathrm{Hom}(G, C)] \subset B^N \subset R^K[\mathrm{Hom}(G, C)][Q].$$

Since the elements of $Q$ are free over $B^N$, this implies that

$$B^N = R^K[\mathrm{Hom}(G, C)].$$

Now let $B$ be the normal basic subalgebra of $R$ constructed from the nucleus $K$ as in Section 3. $B = R^K[u_1, \cdots, u_n]$, and $(u_{m+1}, \cdots, u_n)$ is a $C$-basis for $\mathrm{Hom}(G, C)$. Hence $B = R^K[\mathrm{Hom}(G, C)][u_1, \cdots, u_m]$. Let $Z$ denote the center of $N$. The elements $u_1, \cdots, u_m$ were defined with reference to a basis $x_1, \cdots, x_m$ of the Lie algebra of $N$. Choose this basis so that $x_1, \cdots, x_p$ is a basis for the Lie algebra of $Z$. Then it is clear from the definition of the functions $u_i$ that $u_i \in B^Z$, for every $i > p$. Hence

$$B = B^Z[u_1, \cdots, u_p].$$

If we consider the natural action of the Lie algebra of $Z$ on $B$, we see immediately that, for $i = 1, \cdots, p$, $x_i$ annihilates $B^Z$, while $x_i(u_j) = \delta_{ij}$. Hence we see from the familiar partial differentiation argument that the monomials in the functions $u_1, \cdots, u_p$ are free over $B^Z$. Moreover, if we examine the

argument of Section 3 with which we showed that $S$ is stable under the left translations, we see that, actually, the free $B^Z$-module

$$B^Z + B^Z u_1 + \cdots + B^Z u_p$$

is stable under the left translations.

Now we are in a position to prove the following theorem.

THEOREM 4.1. *Let $G$ be an analytic group, $R$ the algebra of the representative functions on $G$. Let $A$ and $B$ be two normal basic subalgebras of $R$ such that $A_s = B_s$. Then there is an element $x \in N$ such that $B \cdot x = A$.*

*Proof.* We make an induction on the dimension of $N$. Let $Z$ be the center of $N$. If $Z = N$ we have $A^Z = B^Z$, by Lemma 4.1. If $Z \neq N$ we consider $G/Z$, identifying $R(G/Z)$ with $R^Z$. Then $B^Z$ and $A^Z$ become identified with normal basic subalgebras of $R(G/Z)$, and $(B^Z)_s = B_s = A_s = (A^Z)_s$. The radical of $(G/Z)'$ is evidently $N/Z$. Hence, assuming that the theorem is proved in lower dimensions (to start the induction, note that if $N$ is trivial then $A = B$, by Lemma 4.1), there is an element $x \in N$ such that $(B^Z) \cdot x = A^Z$. Replacing $A$ by $A \cdot x^{-1}$, we may therefore assume that $B^Z = A^Z$. Moreover, it evidently suffices to prove the theorem in the case where $B$ is as described above, which we shall now assume.

Every element $f \in R$ may be written uniquely in the form

$$f = \sum_{q \in Q} a_q(f) q,$$

with $a_q(f) \in A$. Then $a_1$ is evidently a $G$-module homomorphism (but not necessarily an algebra homomorphism) of $R$ into $A$, and $a_1$ is the identity map on $B^Z = A^Z$. Let $\alpha$ denote the restriction of $a_1$ to $B^Z + B^Z u_1 + \cdots + B^Z u_p$. Since the monomials in $u_1, \cdots, u_p$ are free over $B^Z$, we can extend $\alpha$ to an *algebra* homomorphism $\beta \colon B \to A$ which, like $\alpha$, commutes with the left translations. We extend $\beta$ to an algebra endomorphism $\gamma$ of $R$ such that $\gamma(q) = q$, for every $q \in Q$. Clearly, $\gamma$ still commutes with the left translations. Furthermore, since $\mathrm{Hom}(G, C) \subset B^Z$, $\gamma$ leaves the elements of $\mathrm{Hom}(G, C)$ fixed. Hence $\gamma(\exp(f)) = \exp(\gamma(f))$, for every $f \in \mathrm{Hom}(G, C)$. Finally, we note that, in the real case, $a_1$ commutes with the complex conjugation, whence also $\gamma$ commutes with the complex conjugation.

In the complex case, it follows at once from [4, Th. 5.1] (with left and right translations interchanged) that $\gamma$ is the right translation by an element $x \in G$. Since $\gamma$ leaves the elements of $B^Z$ fixed, we have, in particular, $f \cdot x = f$, for every $f \in R^K[\mathrm{Hom}(G, C)] = B^N$. From this we conclude first that $x \in K$ (because $R^K$ separates the elements of $G/K$) and then that $x \in N$ (because $\mathrm{Hom}(G, C)$ separates the elements of $K/N$).

In the real case, we appeal to [3, Th. 5.1] to conclude that there is an element $x$ in the universal complexification $G^+$ of $G$ such that $\gamma(f) = f\cdot x$, for every $f \in B$. Now $G^+$ contains the universal complexification $N^+$ of $N$, and it follows as just above, from the fact that $\gamma$ leaves the elements of $R^K[\mathrm{Hom}(G, C)]$ fixed, that $x \in N^+$. Finally, since $\gamma$ commutes with the complex conjugation, it follows that $x \in N$.

Thus, in either case, there is an element $x \in N$ such that $B\cdot x \subset A$. Since both $B\cdot x$ and $A$ are basic subalgebras of $R$, this implies that $B\cdot x = A$, so that our theorem is proved.

Theorems 3.1 and 4.1 give a one to one correspondence between the set of nuclei of $G$ and the set of right $G$-orbits of normal basic subalgebras of $R$. In particular, the two-sidedly $G$-stable subalgebras of $R$ that are generated by the normal basic subalgebras associated with a given nucleus $K$ of $G$ all coincide with one and the same two-sidedly $G$-stable finitely generated subalgebra of $R$, which is thus invariantly associated with the nucleus $K$. In the general case, the representation-theoretical significance of this is not clarified. Moreover, this correspondence is not reversible; the same two-sidedly stable subalgebra may be associated with several, even non-isomorphic, nuclei. This is shown by the following example.

Let $G$ be the group of 7-tuples $(a, b, c, z, r, s, t)$, where $z, r, s, t$ are arbitrary complex numbers, $a, b, c$ are non-zero complex numbers, and the multiplication is given by

$$(a, b, c, z, r, s, t)(a', b', c', z', r', s', t') \\ = (aa', bb', cc', z + z', a'r + r', b's + s', c't + t').$$

For every integer $n$, we define a nucleus $K_n$ of $G$; $K_n$ consists of the elements $(\exp(z), \exp(-z), \exp(nz), z, r, s, t)$, where $z, r, s, t$ range over all complex numbers. We define the functions $\alpha, \beta, \gamma, \zeta, \rho, \sigma, \tau$ on $G$ by $\alpha(a, b, c, z, r, s, t) = a$, etc. Let $q = \exp(\zeta)$. It is easily seen that $R^{K_n}$ is generated by the functions $\alpha q^{-1}, \beta q, \gamma q^{-n}$ and their reciprocals, and that $R^{K_n}[\zeta, \rho, \sigma, \tau]$ is a normal basic subalgebra of $R$; in fact, it results from the construction of Section 3. Now one verifies easily that the two-sidedly stable subalgebra generated by this normal basic subalgebra is $C[\alpha, \beta, \gamma, q, q^{-1}, \alpha^{-1}, \beta^{-1}, \gamma^{-1}, \zeta, \rho, \sigma, \tau]$. This is the same for all $n$. On the other hand, if $n \neq 0$ then $K_n$ is not isomorphic with $K_0$, because $(K_n)'$ is of dimension 3 while $(K_0)'$ is of dimension 2.

**5. The unipotent hull.** For our present purpose, it will be convenient to assume, to begin with, that $G$ is a *complex* analytic group. Let $A$ be the group of all proper automorphisms of $R$, and let $U$ denote the kernel of the

natural representation of $A$ on $R_s$. We shall call $U$ the ***unipotent hull*** of $G$.

We claim that, for every finite dimensional left $G$-stable (and hence also $A$-stable) subspace $S$ of $R$, the natural representation of $U$ on $S$ is unipotent. In order to see this, let us consider a composition series $(0) = S_0 \subset S_1 \subset \cdots \subset S_k = S$ for $S$ as a $G$-module. Let $t_i$ be a linear function on $S$ that vanishes on $S_{i-1}$, and let $s_i \in S_i$. Then the function $t_i/s_i$, where $(t_i/s_i)(x) = t_i(x \cdot s_i)$, for every $x \in G$, is a representative function associated with the representation of $G$ on $S_i/S_{i-1}$, and hence belongs to $R_s$. Hence $u(t_i/s_i) = t_i/s_i$, for every $u \in U$, whence $t_i(u(s_i)) = t_i(s_i)$, for every $u \in U$. Since this holds for all $t_i$ that vanish on $S_{i-1}$ and since $S_i$ is $U$-stable, this implies that $u(s_i) - s_i \in S_{i-1}$, for every $u \in U$. Hence $S$ is unipotent as a $U$-module.

We may choose a finite dimensional two-sidedly stable subspace $S$ of $R$ such that $S$ and the elements of $Q$ generate $R$ and the representation of the subgroup $GU$ of $A$ on $S$ is faithful. This is done as follows: let $f_1, \cdots, f_n$ be a maximal set of linearly indpendent elements of $\mathrm{Hom}(G, C)$, and put $f_0 = c_1 f_1 + \cdots + c_n f_n$, where $c_1, \cdots, c_n$ are rationally independent complex numbers. Put $q_i = \exp(f_i)$. Then, if $K$ is any nucleus of $G$, $q_0, \cdots, q_n$ separate the elements of $K/N$. There is a finite subset $T$ of $R$ containing a set of generators of $R^K$ and such that $T \cup Q$ generates $R$. We let $S$ be the smallest two-sidedly stable subspace of $R$ that contains $\mathrm{Hom}(G, C)$, $T$, and $q_0, \cdots, q_n$. We claim that $S$ satisfies our requirements.

There remains only to show that the representation of $GU$ on $S$ is faithful. Suppose that $x \in G$, $u \in U$, and $xu$ leaves the elements of $S$ fixed. Write $G = H \cdot K$, with $H$ reductive, and $x = hk$, with $h \in H$ and $k \in K$. We have $xu(q_i) = x \cdot q_i = q_i(x) q_i = q_i(k) q_i$. Since $q_i \in S$, we have $xu(q_i) = q_i$. Hence we conclude that $q_i(k) = 1$, for $i = 0, 1, \cdots, n$. By the choice of the $q_i$, this implies that $k \in N$. Since $N \subset U$, this means that we may now assume that $x \in H$. If $f \in R^K$ then $f \in R_s$, so that $xu(f) = x \cdot f$. On the other hand, $f$ belongs to the algebra generated by $S$, whence $xu(f) = f$. Thus $x \cdot f = f$, for every $f \in R^K$. But this implies, since $x \in H$, that $x = 1$. Now $\mathrm{Hom}(G, C) \subset S$, so that $u(f) = f$, for every $f \in \mathrm{Hom}(G, C)$. Since $\exp(f) \in R_s$, we have $u(\exp(f)) = \exp(f)$. By [4, Th. 5.1], it follows that $u$ is the left translation by an element of $G$. Since $S$ and the elements of $Q$ generate $R$, this implies that $u = 1$, q. e. d.

Now let $A_S$ and $U_S$ denote the restrictions to $S$ of $A$ and $U$. By [2, Props. 2.6 and 2.9], $A_S$ is the algebraic group hull of $G_S$. We claim that $U_S$ coincides with the kernel, $V_S$ say, of the semisimple representation associated with the representation of $A_S$ on $S$.

In order to see this, we consider the family of all finite dimensional two-sidedly stable subspaces $T$ of $R$. By the standard decomposition theorem for algebraic linear groups [5, Th. 6.1], the algebraic group $A_T$ is a semidirect product $M_T \cdot V_T$, where $M_T$ is fully reducible. If $T_1 \subset T_2$, the restriction from $T_2$ to $T_1$ is a rational group *epimorphism* $\rho_{T_1,T_2}$ of $A_{T_2}$ onto $A_{T_1}$. By [5, Prop. 3.2], $\rho_{T_1,T_2}(V_{T_2})$ is unipotent and $\rho_{T_1,T_2}(M_{T_2})$ is fully reducible. Hence $A_{T_1}$ is the semidirect product of these two groups, whence it is clear that $\rho_{T_1,T_2}(V_{T_2}) = V_{T_1}$. Now let $\sigma_{T_1,T_2}$ be the restriction to $V_{T_2}$ of $\rho_{T_1,T_2}$, and consider the inverse system of the rational group epimorphisms $\sigma_{T_1,T_2}$. Evidently, the inverse limit of this system is precisely $U$. Hence we conclude from [2, Prop. 2.8] that $U_T = V_T$, for every $T$, which proves our above claim.

In particular, we have a semidirect decomposition $A_S = M_S \cdot U_S$, where $M_S$ is a fully reducible group of automorphisms of $S$. Now we may identify $GU$ with its image in $A_S$, and then we have $GU = M \cdot U$ (semidirect), where $M = M_S \cap (GU)$. Evidently, $GU$ is a normal subgroup of $A_S$, whence $M$ is a normal subgroup of $M_S$. Hence $M$ is a fully reducible group of automorphisms of $S$. It follows that the action of $M$ on the algebra generated by $S$ is semisimple. Since $S$ and the elements of $Q$ generate $R$, it follows that the action of $M$ on $R$ is semisimple. Thus $M$ is *R-reductive*, in the sense that $R$ is semisimple as an $M$-module.

Now suppose that $G$ is a real analytic group. In this case, we shall define the unipotent hull $U$ of $G$ to be the kernel of the representation on $R_s$ of the group $A_r$ of the *real* proper automorphisms of $R$, i.e., the proper automorphisms that commute with the complex conjugation. Now we choose the space $S$ used above so as to be stable under the complex conjugation, and we consider the inverse system of the $\sigma_{T_1,T_2}$ obtained by admitting only those subspaces $T$ that are stable under the complex conjugation. Let $(A_T)_r$ be the subgroup of $A_T$ consisting of the elements of $A_T$ that commute with the complex conjugation of $T$. Then $(A_T)_r$ is a real algebraic subgroup of the group of all real linear automorphisms of $T$. Since $A_T$ is the algebraic hull of $G_T \subset (A_T)_r$, it follows that the Lie algebra of $A_T$ is the tensor product extension over $C$ of the Lie algebra of $(A_T)_r$. Now $V_T$ is the analytic subgroup of $A_T$ whose Lie algebra is the set of all nilpotent elements of the radical of the Lie algebra of $A_T$. Hence the Lie algebra of $V_T$ is spanned over $C$ by the set of all nilpotent elements of the radical of the Lie algebra of $(A_T)_r$. Hence $(V_T)_r$ is the kernel of the semisimple representation associated with the representation of $(A_T)_r$ on $T$. By considering the corresponding Lie algebra map, we see that $\sigma_{T_1,T_2}$ maps $(V_{T_2})_r$ onto all of $(V_{T_1})_r$. Now it follows from the same inverse limit argument we used above (replacing [2, Prop. 2.8] with [2, Prop. 2.11]) that $(V_T)_r = U_T$.

We may now continue exactly as in the complex case to conclude that $GU$ is a semidirect product $M \cdot U$, where $M$ is an $R$-reductive subgroup of $A_r$.

From now on, $G$ may again be either complex or real. Since the analytic group $M$ has a faithful finite dimensional semisimple analytic representation, $M$ is a direct product $P \times V$, where $V$ is a vector group and $P$ is a reductive analytic group ([2, Th. 7.2] and [4, Th. 4.1]). Now $V \cdot U$ is a simply connected, solvable, normal, closed analytic subgroup of $GU$, and $GU$ is the semidirect product $P \cdot (V \cdot U)$. Hence $P$ is a maximal reductive analytic subgroup of $GU$.

Now we recall that if $L$ is a linear analytic group, $M$ a maximal fully reducible analytic subgroup of $L$, and $T$ any fully reducible analytic subgroup of $L$, then there is an element $t$ of the radical of $L'$ such that $tTt^{-1} \subset M$ (see [5, Th. 4.1]).

Let $K$ be a nucleus of $G$, and write $G = H \cdot K$, with $H$ reductive. Since $(GU)_S$ lies in the algebraic hull of $G_S$, we have $(GU)' = G'$. Hence we may conclude from the general theorem just quoted that there is an element $t \in N$ such that $tHt^{-1} \subset P$. Since $G = tHt^{-1} \cdot K$, it follows that $G \subset MK$.

Let $x \in M$ and $y \in K$. Then $xyx^{-1}y^{-1} \in (GU)' = G'$. On the other hand, $xyx^{-1}y^{-1}$ is contained in the radical of $G$. Since there is a continuous arc of such commutators joining $xyx^{-1}y^{-1}$ to 1, we conclude that $xyx^{-1}y^{-1}$ lies in the connected component of the identity of the intersection of $G'$ with the radical of $G$, and thus lies in the radical $N$ of $G'$. Since $N \subset K$, we have therefore $xyx^{-1} \in K$. In particular, we conclude that $M \cap K$ is a normal subgroup of $M$. Hence $M \cap K$ is $R$-reductive. Since $\mathrm{Hom}(G, C)$ separates the elements of $K/N$, while the representation of $G$ on $C + \mathrm{Hom}(G, C)$ is unipotent, this implies that $M \cap K \subset N$. Thus $M \cap K \subset M \cap U$, so that $M \cap K = (1)$. Hence $MK$ is the semidirect product $M \cdot K$.

We have obtained the following result.

THEOREM 5.1. *Let $G$ be a real or complex analytic group, and let $U$ be the unipotent hull of $G$. Then $GU$ is an analytic group having a faithful finite dimensional analytic representation, and $U$ is a nilpotent, simply connected, normal, closed analytic subgroup of $GU$. There is an $R$-reductive closed analytic subgroup $M$ of $GU$ such that $GU$ is the semidirect product $M \cdot U$. If $K$ is any nucleus of $G$ then $G$ is contained in the semidirect product $M \cdot K$.*

COROLLARY 5.1. *The dimension of the unipotent hull $U$ is equal to the dimension of any nucleus $K$.*

*Proof.* Since $U' \subset G' \cap U = N$, it is clear that $U/N$ is a vector group.

We show first that $\mathrm{Hom}(U/N, C)$ is isomorphic with $\mathrm{Hom}(G, C)$. In doing this, we shall identify $\mathrm{Hom}(U/N, C)$ with the subgroup $\mathrm{Hom}(U, C)^N$ of $\mathrm{Hom}(U, C)$. Let $h \in \mathrm{Hom}(U, C)^N$. Since $[M, U] \subset N$, we can extend $h$ uniquely to a homomorphism of $M \cdot U$ into $C$ that is trivial on $M$. The restriction to $G$ of this homomorphism is an element $h' \in \mathrm{Hom}(G, C)$. Clearly, the map $h \to h'$ is a linear homomorphism of $\mathrm{Hom}(U, C)^N$ into $\mathrm{Hom}(G, C)$. Conversely, given $f \in \mathrm{Hom}(G, C)$, define the map $f^* : U \to C$ by $f^*(u) = u(f)(1)$, for every $u \in U$. Since the representation of $G$ on $(C + \mathrm{Hom}(G, C))/C$ is trivial, the same is true for the representation of $U$, because the image of $U$ lies in the algebraic group hull of the image of $G$. From this, it is easily seen that $f^*$ is a homomorphism, and hence that $f^* \in \mathrm{Hom}(U, C)^N$. Moreover, it is verified directly that $(f^*)' = f$ and $(h')^* = h$. Hence $\mathrm{Hom}(U/N, C)$ is isomorphic with $\mathrm{Hom}\ (G, C)$. Hence we have

$$\begin{aligned}\dim(U/N) &= \dim(\mathrm{Hom}(U/N, C)) \\ &= \dim(\mathrm{Hom}(G, C)) = \dim(\mathrm{Hom}(K/N, C)) = \dim(K/N).\end{aligned}$$

Hence $\dim(U) = \dim(K)$, q. e. d.

Put $H = M \cap G$. Then, since $G \subset M \cdot K$, we have $G = H \cdot K$. Furthermore, $M' \subset G$, so that $M' \subset H$, i. e., $M/H$ is abelian.

Now consider the natural representation of $M/H$ on $R^H$. Since $M$ is $R$-reductive, this is semisimple. It follows that $R^H$ is spanned by the elements $f \in R^H$ such that, for every $x \in M$, $x(f) = \phi(x) f$, with $\phi(x) \in C$. Clearly, $\phi \in \mathrm{Hom}(M, C^*)$, where $C^*$ is the multiplicative group of the non-zero complex numbers. Now $\phi$ may be regarded as an element of $\mathrm{Hom}(GU, C^*)$ that is trivial on $HU$. The restriction to $G$ is trivial on $HN$ and therefore is an element $q \in Q$. Obviously, $\phi$ coincides with $q$ on $GU$, and so on $M$, i. e., $\phi(x) = x(q)(1)$, for every $x \in M$. Hence $fq^{-1} \in R^M$, and we have shown that $R^H \subset R^M[Q]$.

We denote by $f \to f'$ the involution of $R$ defined by $f'(x) = f(x^{-1})$, for every $x \in G$. The last result is equivalent to $(R^H)' \subset (R^M)'[Q]$, and it is clear that $(R^H)'$ is the subalgebra of $R$ consisting of all elements that are fixed under the *right* translations with the elements of $H$. By [2, Prop. 2.4] (taking account of the change in notation), we have $(R^H)' R^K = R$. Hence we conclude that $R^K (R^M)'[Q] = R$.

We claim that the elements of $Q$ are free over $(R^M)'$. Let $f_i$ be linearly independent elements of $(R^M)'$, and suppose, contrary to our claim, that there are elements $q_1, \cdots, q_n$ in $Q$ such that $\sum_{i=1}^{n} f_i q_i = 0$. Then we have $\sum_{i=1}^{n} f_i' q_i^{-1} = 0$. Hence, for all $u \in U$ and $m \in M$,

$$\sum_{i=1}^{n} um(f_i')um(q_i^{-1}) = 0, \text{ i.e.,}$$

$$\sum_{i=1}^{n} u(f_i')m(q_i^{-1}) = 0.$$

Since the elements of $U$ separate the $f_i'$ (for $G \subset UM$, and the elements of $M$ leave the $f_i'$ fixed), we can form linear combinations of the above relations, with varying $u$ and fixed $m$, so as to get $m(q_i^{-1}) = 0$, for all $m \in M$, and each $i$. But this is impossible, because $M$ separates the elements of $Q$. This proves our above claim.

Evidently, $(R^H)'$ contains $Q$ and $(R^M)'$. Since $(R^H)' \subset (R^M)'[Q]$, we have therefore $(R^H)' = (R^M)'[Q]$. Now $R$ is canonically isomorphic with the tensor product of $R^K$ and $(R^H)'$. Hence we may now conclude that the elements of $Q$ are free over $R^K(R^M)'$. Hence $R^K(R^M)'$ is a basic subalgebra of $R$. Evidently, $R^K$ and $(R^M)'$ are stable under the left translations.

We have seen in Section 3 that $R_s = D_s[Q]$, where $D$ is any left stable basic subalgebra of $R$. By Theorem 3.1, we may take $D$ so that $D_s = R^K$. Hence $R_s = R^K[Q]$. Hence $(R^K(R^M)')_s \subset R^K[Q]$. But this implies that $(R^K(R^M)')_s = R^K$. Hence we have the following result.

THEOREM 5.2. *Let $G$, $M$, $K$ be as in Theorem* 5.1. *Then $R^K(R^M)'$ is a normal basic subalgebra of $R$, and $K$ is the associated nucleus of $G$.*

It is clear from Theorem 4.1 that every normal basic subalgebra of $R$ has the form of Theorem 5.2; right translation by $x \in N$ changes $M$ to $x^{-1}Mx$.

**6. The group of the proper automorphisms.** Let $A$ denote the group of all proper automorphisms of $R$, let $K$ be a nucleus of $G$, and let $W$ denote the subgroup of $A$ consisting of all elements of $A$ that leave the elements of $R^KR^M$ fixed. In the complex case, let $P$ stand for the natural image of $G$ in $A$. In the real case, let $P$ stand for the natural image of $G^+$ in $A$. Then $P$ is a closed normal subgroup of $A$. In fact, by [3, Th. 5.1], $P$ is the group of all *perfect* automorphisms of $R$, i.e., the proper automorphisms $\alpha$ such that $\exp(\alpha(f)) = \alpha(\exp(f))$, for every $f \in \mathrm{Hom}(G, C)$. We claim that $A$ is the semidirect product $W \cdot P$. Since $R^KR^M[Q] = R$ and $\mathrm{Hom}(G, C) \subset R^KR^M$, it is clear that $P \cap W = (1)$. Hence it suffices to show that $A = WP$. Let $\phi \in \mathrm{Hom}(Q, C^*)$. Since the elements of $Q$ are free over $R^KR^M$ and since $R^KR^M$ is stable under the right translations, there is one and only one element $\alpha_\phi \in W$ such that $\alpha_\phi(q) = \phi(q)q$, for every $q \in Q$. Clearly, the map $\phi \to \alpha_\phi$ is an isomorphism of $\mathrm{Hom}(Q, C^*)$ onto $W$.

Now let $\beta$ be an arbitrary element of $A$. Let $\beta'$ denote the homomorphism

of $R$ into $C$ that is given by $\beta'(r) = \beta(r)(1)$, for every $r \in R$. For every $q \in Q$, there is one and only one $f \in \mathrm{Hom}(G, C)$ such that $q = \exp(f)$. Hence there is an element $\phi \in \mathrm{Hom}(Q, C^*)$ such that

$$\phi(\exp(f)) = \exp(\beta'(f))[\beta'(\exp(f))]^{-1},$$

for every $f \in \mathrm{Hom}(G, C)$. One verifies directly that the automorphism $\alpha_\phi \circ \beta$ of $R$ is a perfect automorphism, so that $\alpha_\phi \circ \beta \in P$. Hence $\beta \in WP$, so that $A = WP$.

We observe also that the elements of $W$ commute with the elements of $M$, and hence with the elements of the maximal reductive analytic subgroup $H = M \cap G$ of $G$. In the real case, we note that $H^+$ is a maximal reductive analytic subgroup of $G^+$; $G^+ = H^+ \cdot K^+$, and the elements of $W$ still commute with the elements of $H^+$. Hence we may state our result as follows

THEOREM 6.1. *The group $A$ of the proper automorphisms of $R$ is a semidirect product $W \cdot P$, where $P$ is the natural image of $G$ (in the complex case) or of $G^+$ (in the real case) in $A$, and $W$ is isomorphic, via restriction to $Q$, with* $\mathrm{Hom}(Q, C^*)$. *Moreover, the elements of $W$ commute with the elements of some maximal reductive analytic subgroup of $P$.*

Let $S$ be a finite dimensional two-sidedly stable subspace of $R$ such that the natural representation $\sigma$ of $P$ on $S$ is faithful. Let $E(S)$ denote the algebra of all linear endomorphisms of $S$. We know that the natural image $A_S$ of $A$ in $E(S)$ is the algebraic group hull of $P_S = \sigma(P)$. Let $\mathfrak{P}$ denote the Lie algebra of $P$. We may identify $\mathfrak{P}$ with its image $\sigma^\cdot(\mathfrak{P})$ in $E(S)$, where $\sigma^\cdot$ is the differential of $\sigma$. Now let $\alpha$ denote the adjoint representation of $P$ in $E(\mathfrak{P})$. If we identify $\mathfrak{P}$ with $\sigma^\cdot(\mathfrak{P})$ then, for every $p \in P$, $\alpha(p)$ becomes identified with the conjugation $\zeta \rightarrow \sigma(p)\zeta\sigma(p)^{-1}$ in the algebra $E(\sigma^\cdot(\mathfrak{P}))$ of all linear endomorphisms of $\sigma^\cdot(\mathfrak{P})$. Since $A_S$ is the algebraic group hull of $\sigma(P)$, $\sigma^\cdot(\mathfrak{P})$ is stable also under the conjugations with the elements of $A_S$, and the corresponding image of $A_S$ in $E(\sigma^\cdot(\mathfrak{P}))$ is the algebraic group hull of $\alpha(P)$. Transferred back to $E(\mathfrak{P})$, this means the following: the conjugation of $P$ effected by an element of $A$ is an analytic automorphism of $P$ and determines a Lie algebra automorphism of $\mathfrak{P}$. We shall call the resulting representation of $A$ in $E(\mathfrak{P})$ the adjoint representation of $A$ on $\mathfrak{P}$. Our result is that this sends $A$ onto the algebraic group hull of the adjoint group of $P$, i.e., *the adjoint representation of $A$ on the Lie algebra of $P$ sends $A$ onto the algebraic group hull of the adjoint group of $P$.*

It is an immediate corollary that if $Z(P)$ is the centralizer of $P$ in $A$ then $A = Z(P)P$ *if and only if the adjoint group of $P$ is algebraic.*

However, the following example shows that, even when $A = Z(P)P$, $P$ need not be a direct factor in $A$. Let $G$ be the group of all pairs $(a, b)$ of complex numbers, with the multiplication

$$(a, b)(a', b') = (a + a', b + \exp(a)b').$$

A right stable basic subalgebra $B$ of $R$ is generated by the constants and the two functions $u_1$, $u_2$, where

$$u_1(a, b) = b, \text{ and } u_2(a, b) = a.$$

The translates of these functions are as follows:

$$u_1 \cdot (a, b) = b + \exp(a)u_1; \qquad (a, b) \cdot u_1 = u_1 + b\exp(u_2);$$
$$u_2 \cdot (a, b) = a + u_2 = (a, b) \cdot u_2.$$

In this case, a subgroup $W$ of $A$ as in Theorem 6.1 can be described explicitly as follows: for every $\gamma \in \mathrm{Hom}(C, C^*)$, there is a $\gamma^* \in W$ defined by:

$$\gamma^*(b) = b, \text{ for every } b \in B, \text{ and}$$
$$\gamma^*(\exp(cu_2)) = \gamma(c)\exp(cu_2) \text{ for every } c \in C.$$

The functions $\exp(cu_2)$ make up the group $Q = \exp(\mathrm{Hom}(G, C))$. Let $t(a, b)$ denote the left translation by $(a, b)$ on $R$. Then one verifies easily that

$$\gamma^* t(a, b)\gamma^{*-1} = t(a, \gamma(1)b).$$

On the other hand,

$$(a', b')(a, b)(a', b')^{-1} = (a, b' + \exp(a')b - \exp(a)b').$$

Hence the conjugation with $\gamma^*$ on $P$ is the conjugation with $t(a', b')$ if and only if, for all $a$ and $b$,

$$\gamma(1)b = (1 - \exp(a))b' + \exp(a')b,$$

i.e., if and only if $b' = 0$ and $\exp(a') = \gamma(1)$.

In particular, we conclude that $A = Z(P)P$. We shall see, however, that $P$ is not a direct factor in $A$. Indeed, suppose that $P$ is a direct factor in $A$. Then there is a homomorphism $\phi\colon W \to P$ such that $\gamma^*\phi(\gamma^*)^{-1} \in Z(P)$, for every $\gamma \in \mathrm{Hom}(C, C^*)$. Putting $\phi(\gamma^*) = t(a_\gamma, b_\gamma)$, we see from the above that we must have $b_\gamma = 0$ and $\exp(a_\gamma) = \gamma(1)$. The map $\gamma \to a_\gamma$ is therefore a homomorphism $\sigma\colon \mathrm{Hom}(C, C^*) \to C$ such that $\exp(\sigma(\gamma)) = \gamma(1)$, for every $\gamma \in \mathrm{Hom}(C, C^*)$.

For each $a \in C$, define the element $\exp_a$ of $\mathrm{Hom}(C, C^*)$ by $\exp_a(c)$

$=\exp(ac)$, for every $c \in C$. The map $a \to \sigma(\exp_a)$ is an endomorphism $\psi$ of $C$. Since $\exp(\psi(a)) = \exp(a)$, we conclude that $\psi(a) - a$ is an integral multiple of $2\pi i$, for every $a \in C$. Evidently, this implies that $\psi(a) = a$, for every $a \in C$, i.e., $\sigma(\exp_a) = a$. It follows from this that $\mathrm{Hom}(C, C^*)$ is the direct product of the subgroup consisting of the $\exp_a$ and the kernel, $H$ say, of $\sigma$.

If $n$ is any positive integer then, for every $\gamma \in \mathrm{Hom}(C, C^*)$, there is one and only one $\gamma_n \in \mathrm{Hom}(C, C^*)$ such that $(\gamma_n)^n = \gamma$; in fact, $\gamma_n(c) = \gamma(c/n)$, for every $c \in C$. It follows that if $h \in H$ then also $h_n \in H$. Since $h(1) = 1$, for every $h \in H$, we conclude that $h(q) = 1$, for every rational number $q$. Hence every $\gamma \in \mathrm{Hom}(C, C^*)$ coincides on the group $Q$ of the rational numbers with an $\exp_a$. But this is a contradiction. For instance, write $C = Q + D$, where $D$ is a $Q$-subspace of $C$ such that $Q \cap D = (0)$. Define the element $\gamma$ of $\mathrm{Hom}(C, C^*)$ as follows: $\gamma(d) = 1$, for every $d \in D$; $\gamma(q) = 1$, for every rational number $q$ that can be written with an odd denominator; $\gamma(q) = \exp(2\pi i q)$, whenever $q$ can be written with a power of 2 as denominator. Since this $\gamma$ does not coincide with an $\exp_a$ on $Q$, we have reached a contradiction. Thus $P$ is not a direct factor of $A$.

**7. Representations as a closed subgroup of a full linear group.** The existence of a right stable basic subalgebra of $R$ leads to a simple proof of the following result which extends (to the complex case) and sharpens a result due to Goto [1, Th. 9].

THEOREM 7.1. *Let $G$ be a real or complex analytic group, and let $\rho$ be an analytic representation of $G$ with finite dimensional representation space $V$. Then there is a finite dimensional analytic representation $\sigma$ with representation space $W$ such that $V \subset W$ (as a $G$-module) and $\sigma(G)$ is closed in the group of all linear automorphisms of $W$.*

*Proof.* Let $B$ denote a right stable basic subalgebra of $R$. We can find a finite dimensional subspace $S$ of $R$ satisfying the following conditions:

(1) $S$ is two-sidedly stable and, in the real case, $S$ is stable under the complex conjugation;

(2) the elements of $S$, together with the constants, generate a subalgebra of $R$ containing $B$;

(3) $S$ contains $\mathrm{Hom}(G, C)$ and the representative functions associated with the given representation $\rho$;

(4) if $s \in S$, and $s = \sum_{q \in Q} b_q(s) q$, with $b_q(s) \in B$ then every $q \in Q$ for which $b_q(s) \neq 0$ belongs to $S$.

In fact, we can evidently find a finite dimensional subspace $S_1$ of $R$ satisfying conditions (1), (2) and (3). Let $S_2$ be the space of the $C$-linear combinations of the elements of $Q$ that occur with non-zero coefficients in the expressions for the elements of $S_1$ as $B$-linear combinations of elements of $Q$. Since $S_1$ is finite dimensional, so is $S_2$, and if $S = S_1 + S_2$ then $S$ satisfies all the above conditions.

Now consider the representation, $\phi$ say, of $G$ by left translations on $S$. We claim that $\phi(G)$ is closed in the group of all linear automorphisms of $S$. Let $A$ denote the group of all proper automorphisms of $R$, and let $A_S$ be its natural image in the group of all linear automorphisms of $S$. Since $A_S$ is the algebraic group hull of $\phi(G)$, it is closed in the full linear group. Hence it suffices to show that $\phi(G)$ is closed in $A_S$. We define a closed subgroup $T$ of $A_S$ as follows: in the complex case, $T$ consists of all $\beta \in A_S$ such that $\beta(\exp(f)) = \exp(\beta(f))$, whenever $f \in \mathrm{Hom}(G, C)$ and $\exp(f) \in S$; in the real case, $T$ consists of all the elements $\beta \in A_S$ satisfying this condition and commuting with the complex conjugation. Now let $\alpha$ be an element of $A$ whose restriction to $S$ belongs to $T$. We can define an algebra endomorphism $\alpha^*$ of $R$ such that $\alpha^*$ coincides with $\alpha$ on $B$, while $\alpha^*(\exp(f)) = \exp(\alpha(f))$, for every $f \in \mathrm{Hom}(G, C)$. Since $B$ is stable under the right translations, it is clear that $\alpha^*$ commutes with the right translations. Hence [2, Prop. 2.5] $\alpha^*$ is a proper automorphism of $R$, and hence a perfect automorphism of $R$. Moreover, in the real case, $\alpha^*$ evidently commutes with the complex conjugation, so that $\alpha^*$ is a real perfect automorphism. Clearly, because of (4), $\alpha^*$ coincides with $\alpha$ on $S$. Hence we conclude that, in the complex case, $T = \phi(G)$. In the real case, we conclude that $T$ is the restriction image of the group $P_r$ of all real perfect automorphisms. It is clear from [3, Th. 5.1] that $\phi(G)$ is the connected component of the identity in $(P_r)_S$. Thus, in either case, $\phi(G)$ is closed in $T$, and therefore also in the full linear group.

Finally, if $n$ is the dimension of $V$ over $C$, $V$ may be identified with a $G$-submodule of the direct sum of $n$ copies of $S$, by condition (3) and [2, Prop. 2.3]. If we take $W$ to be the direct sum of $n$ copies of $S$ and let $\sigma$ be the representation of $G$ on $W$ obtained from $\phi$ in the natural fashion then $\sigma$ evidently satisfies the requirements of Theorem 7.1.

## 8. Nilpotent nuclei and algebraic structures.

THEOREM 8.1. *Let $G$ be a real or complex analytic group, $B$ a normal basic subalgebra of $R$, $K$ the nucleus of $G$ that is associated with $B$. Then $B$ is two-sidedly stable if and only if $K$ is nilpotent. Moreover, in that case,*

*$B$ coincides with the algebra of all representative functions associated with representations that are unipotent on $K$.*

*Proof.* Suppose first that $B$ is two-sidedly stable. By definition, $K$ is the kernel of the representation of $G$ on $B_s$. Let $S$ be any left stable finite dimensional subspace of $B$. Since $B$ is two-sidedly stable, all the representative functions associated with the representation of $G$ on $S$ belong to $B$. Hence we may apply the argument of the beginning of Section 5 (used there for showing that the representation of $U$ on $S$ is unipotent) to conclude that the representation of $K$ on $S$ is unipotent. Since $B$ is a basic subalgebra of $R$, we may choose $S$ so that the representation of $K$ (even of $G$) on $S$ is faithful. Hence we conclude that $K$ is nilpotent.

Now suppose that $K$ is nilpotent. Let $B_1$ be the normal basic subalgebra of $R$ constructed from $K$ as in Section 3. It has been shown in [3, pp. 304-306] that the nilpotency of $K$ implies that $B_1$ coincides with the algebra of all representative functions associated with representations that are unipotent on $K$. Hence $B_1$ is two-sidedly stable, and it follows from Theorem 4.1 that $B_1 = B$. This completes the proof of Theorem 8.1.

*Remark.* Let $H$ be a maximal reductive analytic subgroup of $G$, and let $U$ be the unipotent hull of $G$. Since $H$ is determined up to a conjugation with an element of $N$, the subgroups $HU$ and $HN$ of $A$ are independent of the particular choice of $H$. Now *suppose that $G$ has a nilpotent nucleus $K$, and let $B$ be the corresponding two sidedly stable basic subalgebra of $R$.* We claim that $B$ *uniquely determines an analytic isomorphism of $HU$ onto $G$ that sends $U$ onto $K$ and leaves the elements of $HN$ fixed.* In order to see this, we consider the natural representations of $HU$ and $G$ by proper automorphisms of $B$. Since $HU$ leaves the elements of $Q$ fixed, it is clear that the representation of $HU$ on $B$ is faithful. On the other hand, we know that the representation of $G$ on $B$ is faithful, and that $K$ is the kernel of the representation of $G$ on $B_s$. Hence the image of $K$ in the group of the proper automorphisms of $B$ must contain the image of $U$. By Corollary 5.1, we have $\dim(K) = \dim(U)$. Hence we conclude that the images of $U$ and of $K$ in the group of the proper automorphisms of $B$ coincide. Evidently, this suffices to establish our claim; the isomorphism between $HU$ and $G$ goes via the group of the proper automorphisms of $B$.

Observe that this result immediately implies Theorem 8.4 below; however, we shall give a more direct proof of Theorem 8.4 later on, because this will lead to an explicit description of the set of all nilpotent nuclei.

Now let $G$ be a complex analytic group, and suppose that $G$ has a faith-

ful complex analytic representation $\rho$ such that $\rho(G)$ is an algebraic linear group. Let $B_\rho$ denote the subalgebra of $R$ consisting of all $f \in R$ such that $f \circ \rho^{-1}$ is a rational function on $\rho(G)$. If $\sigma$ is another such representation of $G$ then $B_\rho = B_\sigma$ if and only if the representation $\sigma \circ \rho^{-1}: \rho(G) \to \sigma(G)$ is a rational representation of $\rho(G)$. In order to see this, we need only observe that the representation $\sigma \circ \rho^{-1}$ is rational if and only if its inverse $\rho \circ \sigma^{-1}$ is rational [2, Lemma 10.2]. We shall call such a subalgebra $B_\rho$ of $R$ an *algebraic structure* of $G$. If $S$ is any finite dimensional left stable subspace of $B_\rho$ whose elements, together with the constants, generate $B_\rho$ then the representation of $G$ on $S$ is a faithful representation of $G$ yielding a faithful rational representation of $\rho(G)$, so that the image is an algebraic group, rationally isomorphic with $\rho(G)$. By [2, Lemma 10.1], such a subspace exists in every $B_\rho$. Thus the subalgebras $B_\rho$ correspond in a 1-1 fashion to the rational isomorphism classes of the faithful representations of $G$ as an algebraic linear group.

We shall see that the algebraic structures of $G$ are precisely the two-sidedly stable basic subalgebras of $R$. This will follow easily from the next theorem.

THEOREM 8.2. *Let $G$ be a complex analytic group, and let $\rho$ be a faithful complex analytic representation of $G$ such that $\rho(G)$ is an algebraic linear group. Let $K$ be the kernel of the semisimple representation associated with $\rho$. Then $K$ is a nilpotent nucleus of $G$, and a complex analytic representation $\sigma$ of $G$ yields a rational representation $\sigma \circ \rho^{-1}$ of $\rho(G)$ if and only if $\sigma$ is unipotent on $K$.*

*Proof.* Let $\rho'$ denote the semisimple representation associated with $\rho$. Clearly, $\rho' \circ \rho^{-1}$ is a rational representation of the algebraic group $\rho(G)$. Hence its kernel, $\rho(K)$, is an algebraic subgroup of $\rho(G)$. Since $\rho(K)$ is unipotent, this implies that $\rho(K)$ is connected, simply connected, and nilpotent. Hence $K$ is connected, simply connected, and nilpotent. On the other hand, $\rho'(G)$ is a fully reducible linear algebraic group, and it follows from [4, Th. 4.1] that $\rho'(G)$ is therefore a reductive complex analytic group. Since $G/K$ is isomorphic with $\rho'(G)$, it is now clear that $K$ is a nilpotent nucleus of $G$.

Now write $G$ as a semidirect product $H \cdot K$, with $H$ reductive. Consider the corresponding decomposition $\rho(G) = \rho(H) \cdot \rho(K)$. Let $V$ be the representation space of $\rho$, and let $(0) = V_0 \subset \cdots \subset V_n = V$ be a composition series of $V$. Since $\rho(H)$ is fully reducible, there is a $\rho(H)$-module isomorphism $\phi: \sum_{i=1}^{n} V_i/V_{i-1} \to V$. Consider the algebra isomorphism $\psi: e \to \phi \circ e \circ \phi^{-1}$

of $E(\sum_{i=1}^{n} V_i/V_{i-1})$ onto $E(V)$. The representation of $G$ on $\sum_{i=1}^{n} V_i/V_{i-1}$ is the semisimple representation $\rho'$ associated with $\rho$, and $\rho' \circ \rho^{-1}$ is the rational representation of $\rho(G)$ on $\sum_{i=1}^{n} V_i/V_{i-1}$. Now $\psi \circ \rho' \circ \rho^{-1}$ is precisely the projection of $\rho(G)$ onto $\rho(H)$ that corresponds to the decomposition $\rho(G) = \rho(H) \cdot \rho(K)$. Hence it is clear that this projection is a rational group epimorphism of the algebraic group $\rho(G)$ onto the algebraic group $\rho(H)$. It follows that the projection of $\rho(G)$ onto $\rho(K)$ is also a rational map (though not necessarily a group homomorphism).

Now let $\sigma$ be a complex analytic representation of $G$. If $\sigma \circ \rho^{-1}$ is a rational representation of $\rho(G)$ then it is unipotent on $\rho(K)$, by [5, Prop. 3. 2]. Thus $\sigma$ is unipotent on $K$, in that case. Conversely, suppose that $\sigma$ is unipotent on $K$, and let $f$ be a representative function associated with $\sigma$. It follows from the elementary theory of representative functions (see [2, Section 2]) that there are representative functions $u_i$ on $H$ and representative functions $v_i$ on $K$ such that the $u_i$ are associated with the restriction of $\sigma$ to $H$, the $v_i$ are associated with the restriction of $\sigma$ to $K$, and $f(hk) = \sum_{i=1}^{n} u_i(h)v_i(k)$, for all $h \in H$ and $k \in K$. The functions $u_i \circ \rho^{-1}$ are analytic representative functions on $\rho(H)$ and, since $\rho(H)$ is reductive, they are rational functions on $\rho(H)$, by [4, Th. 5. 2]. Since $\sigma$ is unipotent on $K$, the restriction of $\sigma \circ \rho^{-1}$ to $\rho(K)$ is a rational representation of the unipotent algebraic group $\rho(K)$. Hence the functions $v_i \circ \rho^{-1}$ are rational functions on $\rho(K)$. Since the projections of $\rho(G)$ onto $\rho(H)$ and $\rho(K)$ are rational maps, we may now conclude that $f \circ \rho^{-1}$ is a rational representation of $\rho(G)$. This completes the proof of Theorem 8. 2.

It is clear from Theorems 8. 1 and 8. 2 that each algebraic structure $B_\rho$ is a two-sidedly stable basic subalgebra of $R$, and that the associated nucleus $K$ is the kernel of the semisimple representation associated with $\rho$. Conversely, let $B$ be any two-sidedly stable basic subalgebra of $R$. It is easily seen, as in our proof of Theorem 6. 1, that every proper automorphism of $R$ coincides on $B$ with a left translation by an element of $G$. Let $S$ be a finite dimensional two-sidedly stable subspace of $B$ whose elements, together with the constants, generate $B$. Let $\rho$ be the representation of $G$ by left translations on $S$. Then $\rho(G) = A_S$ and hence, by [2, Props. 2. 6 and 2. 9], $\rho(G)$ is an algebraic subgroup of the group of all linear automorphisms of $S$. Now $B \subset B_\rho$, and since $B_\rho$ is a basic subalgebra of $R$ this implies that $B = B_\rho$. Thus we have the following result.

THEOREM 8.3. *Let $G$ be a complex analytic group. Then the algebraic structures of $G$ are precisely the two-sidedly stable basic subalgebras of $R$. This gives a 1-1 correspondence between the algebraic structures and the nilpotent nuclei of $G$, as follows: given an algebraic structure on $G$, the corresponding nilpotent nucleus is the largest normal subgroup of $G$ on which every rational representation of $G$ (belonging to the given algebraic structure) is unipotent; given a nilpotent nucleus of $G$, the rational representations for the corresponding algebraic structure are precisely those complex analytic representations of $G$ which are unipotent on the given nucleus.*

In view of Theorem 8.3, it is of interest to examine the set of the nilpotent nuclei of $G$. A description of this set is made possible by the fact, discovered by B. Kostant, that any two *nilpotent* nuclei of $G$ are conjugate by an analytic automorphism of $G$ (note that the example at the end of Section 4 shows that this is false for non-nilpotent nuclei). We observe first that it follows from the conjugacy (by inner automorphisms) of any two maximal reductive analytic subgroups of an analytic group that, if $K$ is a nucleus of $G$ and $H$ is *any* maximal reductive analytic subgroup of $G$, then $G$ is the semidirect product $H \cdot K$.

THEOREM 8.4 (B. Kostant). *Let $G$ be a real or complex analytic group, and suppose that $K$ and $L$ are two nilpotent nuclei of $G$. Let $H$ be a maximal reductive analytic subgroup of $G$, so that $G = H \cdot K = H \cdot L$. Then there is an analytic automorphism $\alpha$ of $G$ such that $\alpha$ leaves the elements of $HN$ fixed and $\alpha(K) = L$.*

*Proof.* Let $\mathfrak{G}$ be the Lie algebra of $G$, $\mathfrak{M}$ the maximum nilpotent ideal of $\mathfrak{G}$, $\mathfrak{H}$ the Lie algebra of $H$, $\mathfrak{K}$ and $\mathfrak{L}$ the Lie algebras of $K$ and $L$. We have $\mathfrak{G} = \mathfrak{H} + \mathfrak{L}$, the sum being semidirect, and $\mathfrak{L} \subset \mathfrak{M}$. Hence $\mathfrak{M} = \mathfrak{H} \cap \mathfrak{M} + \mathfrak{L}$.

Under the adjoint representation of $G$ on $\mathfrak{G}$, $H$ operates semisimple on $\mathfrak{G}$. Let $M$ be the maximum nilpotent normal analytic subgroup of $G$. Then $H \cap M$ is normal in $H$, and hence operates semisimply on $\mathfrak{G}$. On the other hand, $M$ operates unipotently on $\mathfrak{G}$. Hence we conclude that $H \cap M$ operates trivially on $\mathfrak{G}$. Hence $H \cap M$ lies in the center of $G$ and $\mathfrak{H} \cap \mathfrak{M}$ lies in the center of $\mathfrak{G}$.

For $x \in \mathfrak{K}$, write $x = \zeta(x) + \gamma(x)$, with $\zeta(x) \in \mathfrak{H} \cap \mathfrak{M}$ and $\gamma(x) \in \mathfrak{L}$. Now, if $u \in \mathfrak{G}$, write $u = v + x$, with $v \in \mathfrak{H}$ and $x \in \mathfrak{K}$, and define $f(u) = v + \gamma(x)$. Since $\gamma$ is a linear isomorphism of $\mathfrak{K}$ onto $\mathfrak{L}$, $f$ is a linear automorphism of $\mathfrak{G}$. Since $\mathfrak{H} \cap \mathfrak{M}$ is in the center of $\mathfrak{G}$, we have $[x, y] = [\gamma(x), y]$, for every $x \in \mathfrak{K}$ and every $y \in \mathfrak{G}$, whence $[f(u), y] = [u, y]$, for all $u$ and $y$ in $\mathfrak{G}$. Hence $[f(u_1), f(u_2)] = [u_1, u_2]$, for all $u_1$, $u_2$ in $\mathfrak{G}$. But $[\mathfrak{G}, \mathfrak{G}] \subset \mathfrak{H} + \mathfrak{K} \cap \mathfrak{L}$, because both $\mathfrak{K}$ and $\mathfrak{L}$ must contain the radical of $[\mathfrak{G}, \mathfrak{G}]$. Hence $f$ is the

identity map on $[\mathfrak{G}, \mathfrak{G}]$, so that the above implies that $f$ is a Lie algebra automorphism of $\mathfrak{G}$.

Now let $H^*$ be the universal covering group of $H$. Then the appropriately defined semidirect product $H^* \cdot K$ (in which $H^*$ operates on $K$ via $H$) is the universal covering group of $G$. Hence our automorphism $f$ of $\mathfrak{G}$ defines an analytic automorphism $f^*$ of $H^* \cdot K$. Since $f$ leaves the elements of $\mathfrak{H}$ fixed, $f^*$ leaves the elements of $H^*$ fixed. Hence $f^*$ induces an automorphism $\alpha$ on $G = H \cdot K$ such that $\alpha$ leaves the elements of $H$ fixed and coincides with $f^*$ on $K$. Since $f$ maps $\mathfrak{K}$ onto $\mathfrak{L}$ and coincides with the identity map on $\mathfrak{N}$, it follows that $\alpha(K) = L$ and that $\alpha$ leaves the elements of $HN$ fixed. This completes the proof.

As an immediate consequence of Theorems 8.3 and 8.4, we obtain the following result.

THEOREM 8.5. *Let $S$ and $T$ be two complex linear irreducible algebraic groups, and suppose that $\sigma$ is a complex analytic isomorphism of $S$ onto $T$. Then there exists a complex analytic automorphism $\alpha$ of $S$ such that $\sigma \circ \alpha$ is a rational isomorphism of $S$ onto $T$.*

*Proof.* Let $B(S) \subset R(S)$ and $B(T) \subset R(T)$ be the given algebraic structures of $S$ and $T$, respectively. Then $B(T) \circ \sigma$ is also an algebraic structure of the analytic group $S$. Let $L$ and $K$ be the nilpotent nuclei of $S$ that correspond to $B(T) \circ \sigma$ and $B(S)$, respectively. There is an analytic automorphism $\alpha$ of $S$ such that $\alpha(K) = L$. This implies that $B(T) \circ \sigma \circ \alpha = B(S)$, i.e., that $\sigma \circ \alpha$ is a rational isomorphism of $S$ onto $T$.

It is known from the theory of Abelian varieties that Theorem 8.5 does not extend to general (non-linear) complex algebraic groups.

Now we proceed to describe the set of all nilpotent nuclei of the real or complex analytic group $G$. As before, let $N$ denote the radical of $G'$, and let $M$ be the maximum nilpotent normal analytic subgroup of $G$. Let $\mathfrak{N}$ and $\mathfrak{M}$ be the Lie algebras of $N$ and $M$, respectively. Let $H$ be a maximal reductive analytic subgroup of $G$. We have seen in the proof of Theorem 8.4 that $H \cap M$ lies in the center of $G$. Since the maximal reductive analytic subgroups of $G$ are conjugate under inner automorphisms, it follows that $H \cap M$ is actually independent of the choice of $H$, and thus is a uniquely determined closed central subgroup $P$ of $G$. Let $\mathfrak{P}$ denote the Lie algebra of $P$. We denote by $\mathrm{Hom}(\mathfrak{M}/(\mathfrak{P} + \mathfrak{N}), \mathfrak{P})$ the space of all linear maps of $\mathfrak{M}$ into $\mathfrak{P}$ that send $\mathfrak{P} + \mathfrak{N}$ onto $(0)$. Finally, let $T$ denote the radical of $G$.

THEOREM 8.6. *Let $G$ be a real or complex analytic group. Then $G$ has a nilpotent nucleus if and only if $T/M$ is reductive. In that case, the set*

*of all nilpotent nuclei of $G$ has the structure of an affine space, with* $\mathrm{Hom}(\mathfrak{M}/(\mathfrak{P}+\mathfrak{N}),\mathfrak{P})$ *as the underlying vector group.*

*Proof.* In the real case, [3, Th. 5.3] implies that there is a two-sidedly stable basic subalgebra of $R$ if and only if $T/M$ is reductive. In the complex case, the same result is contained in [4, Th. 5.4]. Hence the first statement of Theorem 8.6 follows from Theorem 8.1.

Now let $K$ be a nilpotent nucleus of $G$, and write $G=H\cdot K$, as before. If $\mathfrak{K}$ is the Lie algebra of $K$ then, as we have seen in the proof of Theorem 8.4, $\mathfrak{M}$ is the semidirect sum $\mathfrak{P}+\mathfrak{K}$. Now let $\phi\in\mathrm{Hom}(\mathfrak{M}/(\mathfrak{P}+\mathfrak{N}),\mathfrak{P})$. Let $\mathfrak{L}$ be the subspace of $\mathfrak{M}$ consisting of the elements $\phi(x)+x$, where $x$ ranges over $\mathfrak{K}$. Noting that $\mathfrak{P}$ lies in the center of $\mathfrak{G}$ and that $\phi$ annihilates $[\mathfrak{G},\mathfrak{K}]$, we see that $\mathfrak{L}$ is an ideal of $\mathfrak{G}$. Clearly, $\mathfrak{H}+\mathfrak{L}=\mathfrak{H}+\mathfrak{K}=\mathfrak{G}$, and $\mathfrak{H}\cap\mathfrak{L}=(0)$. The proof of Theorem 8.4 shows that there is an analytic automorphism $\alpha$ of $G$ such that $\alpha$ leaves the elements of $H$ fixed and $\alpha(K)$ is the analytic subgroup $L$ of $G$ whose Lie algebra is $\mathfrak{L}$. Thus $L$ is a nilpotent nucleus of $G$. We write $L=\phi\cdot K$. One checks immediately from the definition that, if $\psi$ is any other element of $\mathrm{Hom}(\mathfrak{M}/(\mathfrak{P}+\mathfrak{N}),\mathfrak{P})$, then $\phi\cdot(\psi\cdot K)=(\phi+\psi)\cdot K$, and that $0\cdot K=K$. Moreover, $\phi\cdot K=K$ implies that $\phi=0$, because $\mathfrak{M}=\mathfrak{P}+\mathfrak{K}$. Finally, it is clear from the proof of Theorem 8.4 (where we introduced the map $\zeta$) that, given $K$ and $L$, there is an element $\phi$ in $\mathrm{Hom}(\mathfrak{M}/(\mathfrak{P}+\mathfrak{N}),\mathfrak{P})$ such that $\phi\cdot L=K$. This completes the proof of Theorem 8.6.

In particular, we note that $G$ has only one nilpotent nucleus if and only if either $\mathfrak{P}=(0)$ or $\mathfrak{P}+\mathfrak{N}=\mathfrak{M}$. The first alternative means that $M$ is a nucleus. The second alternative means that $N$ is a nucleus, i.e., that $G/G'$ is reductive (see [4, Th. 5.2] and [2, Ths. 11.1 and 9.1]).

It is clear from Theorem 8.3 that, if $G$ is a complex analytic group, Theorem 8.6 describes the algebraic structures of $G$, simply by reading 'algebraic structure' for 'nilpotent nucleus,' throughout.

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## LINEARE GRUPPEN ÜBER LOKALEN RINGEN.*

Von Wilhelm Klingenberg.

### 1. Resultate.†

1.1. Wir betrachten einen (kommutativen) lokalen Ring $L$. Das grösste Ideal von $L$ wird mit $I$ bezeichnet. Dan ist $L^* = L - L$ eine Gruppe unter der Multiplikation. Falls $J$ ein Ideal in $L$ ist, $J \neq L$, so ist $L/J$ wieder ein lokaler Ring. Es bezeichne

$$g_J: L \to L/J \tag{1}$$

den natürlichen Homomorphismus von $L$ auf $L/J$.

Unter einem *$n$-dimensionalen Vektorraum über $L$*, $V = V_n(L)$, verstehen wir einen $L$-Modul isomorph zu $L^n$. Unter einem *$m$-dimensionalen Unterraum von $V$* verstehen wir einen Untermodul $U$ von $V$, der direkter Summand ist und isomorph zu $L^m$.

Die *allgemeine lineare Gruppe in $n$ Variablen über $L$*, $GL(n, L)$, ist definniert als die Gruppe der lineare Automorphismen von $V = V_n(L)$.

Sei $J$ ein Ideal in $L$. (1) bestimmt den natürlichen Homomorphismus

$$g_J: V_n(L) \to V_n(L/J). \tag{2}$$

Hier lassen wir auch $J = L$ zu; in diesem Falle soll $V_n(L/J)$ den 0-Vektorraum bezeichnen. (2) bestimmt den natürlichen Homomorphismus

$$h_J: GL(n, L) \to GL(n, L/J) \tag{3}$$

mit der Eigenschaft $(h_J\sigma)g_J = g_J\sigma$ für alle $\sigma \in GL(n, L)$. Im Falle $J = L$ soll $GL(n, L/J)$ die Einheitsgruppe $E$ sein. Unter der *Ordnung* $o(X)$ eines Vektors $X \in V_n(L)$ verstehen wir das kleinste Ideal $J$ mit $g_J X = 0$. Unter der *Ordnung* $o(\sigma)$ eines Elements $\sigma \in GL(n, L)$ verstehen wir das kleinste Ideal $J$ so, dass $h_J\sigma \in$ Zentrum $GL(n, L/J)$. Unter der *Ordnung* $o(G)$ einer Untergruppe $G$ von $GL(n, L)$ verstehen wir das kleinste Ideal $J$ so dass $h_J G \subset$ Zentrum $GL(n, L/J)$.

Sei $E_i$, $1 \leqq i \leqq n$, eine Basis von $V$. Wenn, für $X \in V$, $X = \sum E_i x_i$, so ist $o(X)$ gleich dem von den $x_i$, $1 \leqq i \leqq n$, erzeugten Ideal. Wenn, für

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† Die wichtigsten Resultate der Arbeit wurden angekündigt in [6].

$\sigma \in GL(n,L)$, $\sigma E_j = \sum E_i a_{ij}$, so ist $o(\sigma)$ gleich dem von den $a_{ij}$, $i \neq j$, und $a_{ii} - a_{jj}$, $1 \leqq i, j \leqq n$, erzeugten Ideal. Die Ordnung $o(G)$ einer Untergruppe $G$ von $GL(n,L)$ wird erzeugt von den Ordnungen $o(\sigma)$, $\sigma \in G$.

1.2. Sei $J$ ein Ideal in $L$. Unter der *allgemeinen Kongruenzuntergruppe mod $J$ von $GL(n,L)$, $GC(n,L,J)$*, verstehen wir die Gruppe

$$GC(n,L,J) = h_J^{-1}(\text{Zentrum } GL(n,L/J)). \tag{4}$$

Offenbar ist $GC(n,L,J)$ eine invariante Untergruppe von $GL(n,L)$ der Ordnung $J$. Insbesondere haben wir $GC(n,L,L) = GL(n,L)$ und $GC(n,L,0) = \text{Zentrum } GL(n,L)$ isomorph $L^*$. Hier bezeichnet 0 das 0-Ideal in $L$.

Sei $J$ ein Ideal in $L$. Unter der *speziellen Kongruenzuntergruppe mod $J$ von $GL(n,L)$, $SC(n,L,J)$*, verstehen wir diejenige invariante Untergruppe von $GL(n,L)$, die erzeugt wird von den Transvektionen der Ordnung $\subset J$, d.h., von den Transvektionen in $GC(n,L,J)$. Eine *Transvektion* $\tau$ ist dabei ein Element aus $GL(n,L)$, für das es einen Unterraum $H$ der Kodimension 1 (kunz: Hyperebene) in $V$ gibt so, dass $\tau|_H = \text{Identität}$ und so, dass $\tau X - X \in H$ für alle $X \in V$.

Offenbar ist $SC(n,L,J)$ eine invariante Untergruppe von $GL(n,L)$ mit einer Ordnung $\subset J$, die in $GC(n,L,J)$ enthalten ist. Insbesondere ist $SC(n,L,0) = E = \text{Einheitsgruppe}$. Für $SC(n,L,L)$ schreiben wir auch $SL(n,L)$ und nennen diese Gruppe die *spezielle lineare Gruppe in $n$ Variablen über $L$.*

1.3. Lemma. *Zwei Vektoren $A$ und $B$ von $V = V_n(L)$ haben dann und nur dann dieselbe Ordnung $o(A) = o(B)$, wenn es ein Element $\sigma \in GL(n,L)$ gibt so, dass $\sigma A = B$.*

Hieraus folgt

Satz 1. *Zwei Transvektionen $\tau_1, \tau_2$ aus $GL(n,L)$ von derselben Ordnung $o(\tau_1) = o(\tau_2)$ sind konjugiert in $GL(n,L)$.*

Ergänzung. *Falls $n \geqq 3$ und $o(\tau_1) = o(\tau_2)$ ein Hauptideal ist, so sind $\tau_1$ und $\tau_2$ schon konjugiert in $SL(n,L)$.*

1.4. Hiermit lassen sich die speziellen Kongruenzuntergruppen folgendermassen charakterisieren:

Theorem 1. *Sei $J$ ein Ideal in dem lokalen Ring $L$ und sei $G$ eine Untergruppe von $GL(n,L)$. Folgende Aussagen sind äquivalent:*

(a) $G = SC(n,L,J)$.

(b) *$G$ besteht aus den Elementen $\sigma \in GL(n,L)$ mit* $\det\sigma = 1$ *und $h_J\sigma =$ Identität.*

(c) *$G =$ gemischte Kommutatorgruppe* $\mathrm{Komm}(GL(n,L), GC(n,L,J))$. *Für $n = 2$ wird hierbei $L/I \neq \mathbf{F}_2$ vorausgesetzt.*

KOROLLAR. *Sei $G$ eine Untergruppe von $GL(n,L)$. Folgende Aussagen sind äquivalent:*

(a) $G = SL(n,L)$.

(b) *$G$ besteht aus den Elementen $\sigma \in GL(n,L)$ mit* $\det\sigma = 1$.

(c) *$G$ ist die Kommutatorgruppe von $GL(n,L)$. Hier wird für $n = 2$ vorausgesetzt: $L \neq \mathbf{F}_2$.*

1.5. THEOREM 2. *Sei $J$ ein Ideal in dem lokalen Ring $J$.*

(i) *$GC(n,L,J)/SC(n,L,J)$ ist isomorph zu der Untergruppe $U(n,L,J)$ von $L^* \times (L/J)^*$, die gebildet wird von den Elementen $(a,b) \in L^* \times (L/J)^*$ mit $g_J a = b^n$.*

(ii) *Setze $HC(n,L,J) = GC(n,L,J) \cap SL(n,L)$. Dann ist*

$$HC(n,L,J)/SC(n,L,J)$$

*isomorph zur Gruppe $E_n((L/J)^*)$ der $n$-ten Einheitswurzeln in $(L/J)^*$.*

KOROLLAR. *$GL(n,L)/SL(n,L)$ ist isomorph zu $L^*$. Zentrum $GL(n,L)$ ist isomorph zu $L^*$. Zentrum $SL(n,L)$ ist isomorph zu $E_n(L^*)$.*

1.6. $GC(n,L,J)$ und $SC(n,L,J)$ sind invariante Untergruppen der Ordnung $J$. Nach Theorem 2 ist $GC(n,L,J)/SC(n,L,J)$ kommutativ, also ist jede Untergruppe $G$ von $GL(n,L)$, die der Beziehung

$$GC(n,L,J) \supset G \supset SC(n,L,J) \tag{5}$$

genügt, eine invariante Untergruppe der Ordnung $o(\mathrm{G}) \subset J$.

Das Hauptergebnis der vorliegenden Arbeit ist nun, dass umgekehrt jede invariante Untergruppe $G$ der Ordnung $o(G) = J$ der Beziehung (5) genügt. Wir werden sogar allgemeiner beweisen, dass dies für die unter $SL(n,L)$ invarianten Untergruppen $G$ von $GL(n,L)$ gilt.

Zunächst beweisen wir

SATZ 2. *Sei $(\tau_\alpha)_{\alpha \in A}$ eine Menge von Transvektionen $\tau_\alpha$ der Ordnung $o(\tau_\alpha) = J_\alpha$. Die von den $\tau_\alpha$ erzeugte, unter $SL(n,L)$ invariante Untergruppe*

*$G$ in $GL(n,L)$ ist gleich $SC(n,L,J)$, wo $J$ das von den $J_\alpha$, $\alpha \in A$, erzeugte Ideal in $L$ ist. Für $n=2$ setzen wir voraus:* $\mathrm{char}(L/I) \neq 2$.

Sodann beweisen wir den fundamentalen

Satz 3. *Sei $G$ eine unter $SL(n,L)$ invariante Untergruppe von $GL(n,L)$ der Ordnung $o(G)=J$. Dann enthält $G$ die Gruppe $SC(n,L,J)$. Hier setzen wir für $n=2$ voraus:* $\mathrm{char}(L/I) \neq 2$ *und* $L/I \neq \mathbf{F}_3$.

1.7. Durch Zusammenfassung der vorstehenden Ergebnisse erhalten wir den folgenden Satz über die Struktur der allgemeinen und der speziellen linearen Gruppe über einem lokalen Ring $L$:

Theorem 3. *Sei $L$ ein lokaler Ring.*

(i) *Eine Untergruppe $G$ von $GL(n,L)$, die invariant ist unter $SL(n,L)$, bestimmt ein Ideal $J$ von $L$ so, dass (5) gilt. Umgekehrt ist jede Untergruppe $G$ von $GL(n,L)$, die den Beziehungen (5) genügt, eine invariante Untergruppe von $GL(n,L)$ der Ordnung $o(G)=J$.*

(ii) *Eine invariante Untergruppe $G$ von $SL(n,L)$ bestimmt ein Ideal $J$ von $L$ so, dass*

$$HC(n,L,J) = GC(n,L,J) \cap SL(n,L) \supset G \supset SC(n,L,J) \tag{6}$$

*gilt. Umgekehrt ist jede Untergruppe $G$ von $SL(n,L)$, die den Beziehungen (6) genügt, invariant und von der Ordnung $J$.*

*Für $n=2$ setzen wir voraus:* $\mathrm{char}(L/I) \neq 2$ *und* $L/I \neq \mathbf{F}_3$.

Korollar. (Dieudonné [4], [5]) *Sei $L$ ein kommutativer Körper.*

(i) *Für eine Untergruppe $G$ von $GL(n,L)$, die invariant ist unter $SL(n,L)$, gilt eine folgenden Beziehungen:*

$$\begin{aligned} GL(n,L) &\supset G \supset SL(n,L), \\ \text{Zentrum}\, GL(n,L) &\supset G \supset E. \end{aligned} \tag{7}$$

*Umgekehrt ist jede Untergruppe $G$ von $GL(n,L)$, die einer der Beziehungen (7) genügt, invariant in $GL(n,L)$.*

(ii) *Die invarianten Untergruppen $G$ von $SL(n,L)$, $G \neq SL(n,L)$, gehören zu* $(\text{Zentrum}\, GL(n,L)) \cap SL(n,L) = \text{Zentrum}\, SL(n,L)$.

*Hier setzen wir für $n=2$ voraus:* $\mathrm{char}\, L \neq 2$ *und* $L \neq \mathbf{F}_3$.

1.8. Wir können auf Grund von Theorem 2 und 3 folgendes feststellen:

(a) Die unter $SL(n, L)$ invarianten Untergruppen $G$ von $GL(n, L)$ der Ordnung $J$ genügen den Bedingungen (5), das heisst, $GC(n, L, J)$ und $SC(n, L, J)$ sind die grösste bzw. die kleinste unter $SL(n, L)$ invariante Untergruppe der Ordnung $J$ in $GL(n, L)$, und jede zwischen diesen beiden Gruppen gelegene Gruppe $G$ ist invariant in $GL(n, L)$ und von der Ordnung $J$.

(b) Jede unter $SL(n, L)$ invariante Untergruppe $G$ von $GL(n, L)$ (die dann auch invariant ist unter $GL(n, L)$) ist bestimmt durch ihre Ordnung $o(G) = J$ und durch die Gruppe $G/SC(n, L, J)$, die eine Untergruppe der in Theorem 2 eingeführten kommutativen Gruppe $U(n, L, J)$ ist.

Das Entsprechende gilt für die invarianten Untergruppen von $SL(n, L)$.

1.9. Für den Fall $L = \mathbf{Z}/(p^r)$, $p$ eine Primzahl, sind die vorstehenden Ergebnisse bewiesen von Brenner [2].

Für den Fall, dass $L$ ein verallgemeinerter Bewertungsring ist, das heisst, ein kommutativer Ring mit Eins, für den die Ideale total geordnet sind, sind vorstehende Ergebnisse bewiesen in [7].

Wie schon in [7], schliessen wir uns in den Bezeichnungen und den Anordnungen der Beweise der Darstellung an, die Artin [1] von der Theorie der linearen Gruppen gegeben hat.

1.10. Die vorstehenden Ergebnisse bleiben im wesentlichen gültig, wenn man *lineare Gruppen $GL(n, L)$ über nichtkommutativen lokalen Ringen $L$* betrachtet. Ein nichtkommutativer lokaler Ring ist, nach Cartan-Eilenberg [3], ein nichtkommutativer Ring $L$ mit Eins und einem grössten Ideal $I$; $I$ ist zweiseitiges Ideal; $L/I$ est ein nicht notwendig kommutativer Körper. Wir wollen ferner voraussetzen, dass jedes Linksideal $J$ von $L$ auch Rechtsideal ist. Die Klasse dieser Ringe unfasst, für $I = 0$, die nichtkommutativen Körper.

Die im folgenden geführten Beweise bleiben, jedenfalls für $n \geqq 3$, auch für die linearen Gruppen $GL(n, L)$ über einem solchen nichtkommutativen lokalen Ring gültig; nur beim Beweis der Beziehung:

$$SC(n, L, J) = \mathrm{Komm}(GL(n, L), GC(n, L, J)),$$

Theorem 1(c), sind einige zusätzliche Überlegungen nötig, die damit zusammenhängen, dass man in Theorem 1(b) den Begriff der Determinante verfeinern muss. Und zwar tritt an die Stelle der gewöhnlichen Determinante eine für jedes Ideal $J$ von $L$ erklärte $J$-Determinante:

$$\det_J : GC(n, L, J) \to \bar{L}(J, n)$$

wobei $\bar{L}(J,n) = g_J^{-1}(\text{Zentrum}(L/J)^*)^n/\text{Komm}(L^*, g_J^{-1}\,\text{Zentrum}(L/J)^*)$ eine Untergruppe von $L^*/\text{Komm}(L^*)$ ist. Für $J = L$ stimmt $\det_J$ überein mit der von Dieudonné [4], [5] eingeführten Determinante über nichtkommutativen Körpern.

In Theorem 2 ist entsprechend zu lesen: $GC(n,L,J)/SC(n,L,J)$ ist isomorph zu der Untergruppe der Paare $(a,b)$ in $\bar{L}(J,n) \times \text{Zentrum}(L/J)^*$ mit $g_J a = b^n$. Speziell für $J = L$ and $J = 0$ liefert dies: $GL(n,L)/SL(n,L)$ ist isomorph zu $L^*/\text{Komm}\,L^*$, Zentrum $GL(n,L)$ ist isomorph zu Zentrum $L^*$.

Als Korollar erhält man die Sätze von Dieudonné [4], [5] über die Struktur der linearen Gruppen über nichtkommutativen Körpen.

Die vollständigen Beweise dieser Verallgemeinerung sollen anderweitig veröffentlicht werden.

## 2. Beweise.

2.1. *Beweis des Lemmas.* Falls es ein $\sigma \in GL(n,L)$ gibt mit $\sigma A = B$, dann ist $g_J A = 0$ äquivalent mit $g_J B = g_J \sigma A = h_J \sigma g_J A = 0$, also $o(A) = o(\sigma A) = o(B)$.

Sei nun umgekehrt $o(A) = o(B) = J$. Bezüglich einer Basis $E_i$, $1 \leqq i \leqq n$, sei $A = \sum E_i a_i$. $J$ wird erzeugt von den $a_i$. Sei $u_\alpha$, $1 \leqq \alpha \leqq p \leqq n$, eine minimale Anzahl von erzeugenden Elementen von $J$. Dann ist also $u_\alpha = \sum r_{\alpha j} a_j$ und $a_i = \sum b_{i\beta} u_\beta$ und $u_\alpha = \sum r_{\alpha j} b_{j\beta} u_\beta$, das heisst $\sum(\sum r_{\alpha j} b_{j\beta} - \delta_{\alpha\beta}) u_\beta = 0$. Da die $u_\alpha$ ein minimales System von Erzeugenden bilden, kann keines der Elemente $\sum r_{\alpha j} b_{j\beta} - \delta_{\alpha\beta}$ zu $L^*$ gehören, mit anderen Worten, $\sum r_{\alpha j} b_{j\beta} - \delta_{\alpha\beta} = 0 \mod I$.

Wir setzen $F_\alpha = \sum E_j b_{j\alpha}$ und behaupten, dass die $F_\alpha$ linear unabhängig sind $\mod I$. In der Tat, aus $\sum F_\beta c_\beta = \sum E_j b_{j\beta} c_\beta = 0 \mod I$ folgt $\sum b_{j\beta} c_\beta = 0 \mod I$ und also $\sum r_{\alpha j} b_{j\beta} c_\beta = c_\alpha = 0 \mod I$. Die $F_\alpha$ können zu einer Basis $F_i$ ergänzt werden. Denn sei $E_i$, $1 \leqq i \leqq n$, eine Basis von $V$ und $F_\alpha = \sum E_i f_{i\alpha}$. Die $(n,p)$-Matrix $(f_{i\alpha})$, $1 \leqq i \leqq n$, $1 \leqq \alpha \leqq p$, hat $\mod I$ den Rang $p$, sie lässt sich also $\mod I$ zu einer $(n,n)$-Matrix vom Rang $n$ erweitern. Wenn dann $(f_{ij})$, $1 \leqq i, j \leqq n$, ein Urbild einer solchen Erweiterung ist, so setze man $F_j = \sum E_i f_{ij}$. Offenbar ist $A = \sum F_\alpha u_\alpha$.

Da $o(B) = o(A) = J$, können wir auch $B$ in der Form $B = \sum G_\alpha u_\alpha$ schreiben, wo die $G_\alpha$ linear unabhängig sind $\mod I$. Die $G_\alpha$ können also zu einer Basis $G_i$ ergänzt werden. Wir erklären $\sigma \in GL(n,L)$ durch $\sigma F_i = G_i$. Dann wird $\sigma A = B$.

2.2. *Beweis von Satz* 1. Sei $\tau$ eine Transvektion, $H$ eine Hyperebene mit $\tau|_H = \text{Identität}$. Sei $\phi$ eine Linearform mit $\phi^{-1}(0) = H$. Sei $B$ ein

Vektor mit $\phi(B)=1$. Dann ist $X-B\phi(X)\in H$ für alle $X\in V$ und daher $\tau X-\tau B\phi(X)=X-B\phi(X)$. Wir setzen $A=\tau B-B\in H$. Damit wird

$$\tau X=X+A\phi(X). \tag{8}$$

Wir sagen: *Der Vektor $A$ in* (8) *gehört zu der Transvektion $\tau$.* $A$ ist durch $\tau$ (oder genauer: durch $H$) nur bis auf ein Element $c\in L^*$ bestimmt; denn wenn $\phi$ ersetzt durch $c\phi$, dann geht $A$ über in $Ac^{-1}$. Offenbar ist $o(\tau)=o(A)$, wenn $A$ zu $\tau$ gehört.

Wir machen jetzt die Voraussetzungen von Satz 1. $\tau_1$ und $\tau_2$ stellen wir dar durch

$$\tau_r X=X+A_r\phi_r(X) \qquad (r=1,2). \tag{9}$$

$B_r$ sei so gewählt dass $\phi_r(B_r)=1$. Da $o(\tau_1)=o(\tau_2)$, ist $o(A_1)=o(A_2)$. Nach dem Lemma gibt es also ein $\sigma\in GL(n,L)$ mit $\sigma A_1=A_2$. Aus dem Lemma für $n-1$ folgt, dass man zugleich $\sigma H_1=H_2$ erreichen kann, wo $H_r=\phi_r^{-1}(0)$, und dann kann man noch erreichen $\sigma B_1=B_2$. Dann ist jedoch $\tau_2=\sigma\tau_1\sigma^{-1}$. Die Ergänzung zu Satz 1 beweisen wir in 2.4.

2.3. *Beweis von Theorem* 1.

2.3.1. Wir bezeichnen die durch (b) definierte Gruppe mit $H_J$ und die durch (c) definierte Gruppe mit $K_J$.

2.3.2. Offenbar gilt für jede Transvektion $\tau$ mit $o(\tau)\subset J$: $\det\tau=1$ und $h_J\tau=$ Identität, also $SC(n,L,J)\subset H_J$.

Sei nun umgekehrt $\sigma$ ein Element aus $H_J$. Falls $J=L$, das heisst, falls $SC(n,L,J)=SL(n,L)$, dann kann man bekanntlich (vgl. Dieudonné l. c. oder Artin l. c.) eine Matrixdarstellung von $\sigma$ durch Multiplikation mit geeigneten Elementen von $SL(n,L)$ von rechts und von links auf die Form $\mathrm{diag}(1,1,\cdots,a)$ bringen, wo $a=\det\sigma=1$ ist.

Sei jetzt $\sigma\in H_J$ mit $J\subset I$. Sei $\sigma$ dargestellt durch die Matrix $((a_{ik}))$. Da $h_J\sigma=$ Identität, ist $a_{ik}\in J$ für $i\neq k$ und $a_{ii}-1\in J$. Indem wir von links und von rechts mit geeigneten Elementen aus $SC(n,L,J)$ multiplizieren, können wir $((a_{ik}))$ auf die Gestalt $\mathrm{diag}(1+u_1,1+u_2,\cdots,1+u_n)$, $u_i\in J$, bringen.

Die Formel

$$\begin{pmatrix}1 & -(1+u)^{-1}u\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ u(1+u)^{-1} & 1\end{pmatrix} = \begin{pmatrix}(1+u)^{-1} & 0\\ 0 & 1+u\end{pmatrix} \tag{10}$$

zeigt, dass auch das Element $\operatorname{diag}((1+u)^{-1}, 1, \cdots, 1+u)$, $u \in J$, zu $SC(n, L, J)$ gehört. Daher können wir durch Multiplikation mit geeigneten Elementen von $SC(n, L, J)$ aus $((a_{ik}))$ sogar die Matrix $\operatorname{diag}(1, 1, \cdots, 1+w)$, $w \in J$, erhalten. Wegen $\det((a_{ik})) = \det \operatorname{diag}(1, 1, \cdots, 1+w) = 1 + w = 1$ ist $w = 0$, also $\sigma \in SC(n, L, J)$, d. h. $H_J \subset SC(n, L, J)$.

2.3.3. Ein erzeugendes Element von $K_J$ hat die Gestalt $\rho\sigma\rho^{-1}\sigma^{-1}$ mit $\rho \in GL(n, L)$ und $\sigma \in GC(n, L, J)$. Da $h_J\sigma \in$ Zentrum $GL(n, L/J)$, ist $h_J\rho\sigma\rho^{-1}\sigma^{-1} = h_J\rho h_J\sigma h_J\rho^{-1} h_J\sigma^{-1} =$ Indentität. Da ferner $\det \rho\sigma\rho^{-1}\sigma^{-1} = 1$, haben wir $K_J \subset H_J = SC(n, L, J)$.

Sei nun umgekehrt $\tau$ eine Transvektion aus $SC(n, L, J)$. Wir schreiben $\tau$ in der Form (8) mit $A \in \phi^{-1}(0) = H$. Wir betrachten zunächst den Fall $n \geqq 3$. Dann ist also $\dim H \geqq 2$. Wir behaupten, dass wir dann $A$ in der Form $A = A_2 - A_1$ schreiben können mit $o(A_1) = o(A_2) = o(A)$ und $A_1 \in H$, $A_2 \in H$.

In der Tat, es sei $A = \sum E_i a_i$, wo $E_i$, $1 \leqq i \leqq n-1$, eine Basis für $H$ ist. Falls $n - 1 = 2m$, so setzen wir

$$A_1 = \sum E_{2j-1}(a_{2j} - a_{2j-1}) + \sum E_{2j} a_{2j-1},$$

$$A_2 = \sum E_{2j-1} a_{2j} + \sum E_{2j}(a_{2j-1} + a_{2j}) \qquad (j = 1, \cdots, m).$$

Falls $n - 1 = 2m + 1$, so ersetzen wir in den vorstehenden Ausdrücken den Summanden $j = 1$ durch

$$E_1(a_{2m+1} + a_2 - a_1) + E_2(a_1 + a_{2m+1}) + E_{2m+1} a_1,$$

$$E_1(a_2 + a_{2m+1}) + E_2(a_1 + a_2 + a_{2m+1}) + E_{2m+1}(a_{2m+1} + a_1).$$

Wir definieren nun $\tau_r$ $(r = 1, 2)$ durch (9) mit $\phi_r = \phi$. Nach Satz 1 gibt es $\sigma \in GL(n, L)$ mit $\tau_2 = \sigma\tau_1\sigma^{-1}$. Also

$$\tau = \tau_2\tau_1^{-1} = \sigma\tau_1\sigma^{-1}\tau_1^{-1} \in K_J, \text{ d. h. } SC(n, L, J) \subset K_J.$$

Im Falle $n = 2$ benutzen wir die Voraussetzung dass $L/I \neq \mathbf{F}_2$. Es gibt dann ein $c \in L^*$ so, dass $A = A_2 - A_1$, $A_2 = A(1 + c)$, $A_1 = Ac$, $o(A) = o(A_1) = o(A_2)$. Nach Satz 1 gibt es ein $\sigma \in GL(2, L)$ so, dass $\tau = \sigma\tau_1\sigma^{-1}\tau_1^{-1} \in K_J$, also $SC(2, L, J) \subset K_J$.

2.4. *Beweis von Satz* 1, *Ergänzung*. Wir verwenden die Bezeichnungen aus 2.2. Insbesondere ist also $\tau_r$ $(r = 1, 2)$ dargestellt durch (9). Sei $H_r = \phi_r^{-1}(0)$. Nach Voraussetzung ist $o(\tau_r) = o(A_r) = (a)$ ein Hauptideal. Das heisst, wir können $A_r$ in der Form $A_r = E_r a$, $o(E_r) = L$, schreiben. Da $n \geqq 3$, also $\dim H_r = n - 1 \geqq 2$, gibt es in $H_r$ einen Vektor $F_r$, der linear

unabhängig mod $I$ von $E_r$ ist, $r=1,2$. Wir ergänzen $E_1$, $F_1$ zu einer Basis von $H_1$ und ergänzen, für ein beliebiges $c \in L^*, E_2, F_2c$ zu einer Basis von $H_2$. Damit erkennen wir, dass es stets $\sigma \in GL(n,L)$ so gibt, dass $\sigma E_1 = E_2$ (also $\sigma A_1 = A_2$), $\sigma F_1 = F_2 c$ und $\sigma H_1 = H_2$, $\sigma B_1 = B_2$, also $\tau_2 = \sigma\tau_1\sigma^{-1}$. Wenn wir $c \in L^*$ geeignet wählen, können wir erreichen, dass $\det \sigma = 1$, das heisst, $\sigma \in SL(n,L)$.

2.5. *Beweis von Theorem* 2. Wir betrachten den Homomorphismus

$$f\colon \sigma \in GC(n,L,J) \to (\det\sigma, h_J\sigma) \in L^* \times \text{Zentrum}\, GL(n,L/J).$$

Die rechts stehende Gruppe ist isomorph $L^* \times (L/J)^*$. Nach Theorem 1 ist $\text{Kern}(f) = SC(n,L,J)$.

Sei $\sigma \in GC(n,L,J)$ und $\det\sigma = a \in L^*$. Dann ist

$$g_J a = g_J \det\sigma = \det h_J\sigma = \det \operatorname{diag}(b,b,\cdots,b) = b^n, \quad b \in (L/J)^*.$$

Zu einem $b \in (L/J)^*$ wähle man ein $b' \in L^*$ mit $g_J b' = b$. Dann stellt die Matrix $((b'\delta_{ik}))$ ein Element $\sigma$ aus $GC(n,L,J)$ dar mit $g_J \det\sigma = b^n$.

2.6. *Beweis von Satz* 2.

2.6.1. Wir betrachten zunächst den Fall $n \geqq 3$ und nehmen an, das die Menge $(\tau_\alpha)_{\alpha\in A}$ aus einem einzigen Element $\tau$ besteht. $\tau$ sei dargestellt durch (8) mit $A = \sum E_i a_i$, $1 \leqq i \leqq n-1$, wobei die Vektoren $E_i$, $1 \leqq i \leqq n-1$, eine Basis für die Hyperebene $H = \phi^{-1}(0)$ bilden. $J = o(\tau) = o(A)$ wird erzeugt von den Elementen $a_i$, $1 \leqq i \leqq n-1$.

Wir definieren $A_1$ und $A_2$ durch

$$A_1 = \sum_1^{n-1} E_i a_i - \sum_2^{n-1} E_i a_{i-1}; \qquad A_2 = \sum_2^{n-1} E_i a_{i-1}.$$

Wir haben $A_2 = A - A_1$ und $o(A) = o(A_1)$. Wir erklären die Transvektionen $\tau_r$ $(r=1,2)$ durch (9). Nach Satz 1 gibt es $\sigma \in GL(n,L)$ mit $\tau_1 = \sigma\tau\sigma^{-1}$. Da man speziell $\sigma$ so wählen kann, dass $\sigma E_i = E_i - E_{i+1}$ $(i < n-1)$, $\sigma E_{n-1} = E_{n-1}$, erkennt man, dass es sogar in $SL(n,L)$ ein $\sigma$ gibt mit $\tau_1 = \sigma\tau\sigma^{-1}$, d.h. $\tau_1 \in G =$ die von $\tau$ erzeugte unter $SL(n,L)$ invarianten Untergruppe von $GL(n,L)$. Dann auch $\tau_2 = \tau\tau_1^{-1} \in G$, und indem wir auf $\tau_2$ einen entsprechenden Schluss anwerden erkennen wir, dass $G$ eine Transvektion enthält mit zugehörigem Vektor $E_{n-1}a_1$.

Auf Grund der Ergänzung zu Satz 1 können wir sagen, dass $G$ alle Transvektionen enthält mit zugehörigem Vektor $Ea_1c(=(Ec)a_1)$, wo $o(E) = L$ und $c \in L^*$. Offenbar ergibt sich genau so, dass $G$ auch die Transvek-

tionen mit zugehörigem Vektor $Ea_ic$, $1 \leqq i \leqq n-1$, $o(E) = L$, $c \in L^*$, enthält. Wenn $c \in I$, so $c-1 \in L^*$. Da $Ea_ic = Ea_i(c-1) + Ea_i$, enthält $G$ auch die Transvektionen mit Vektor $Ea_ic$, $o(E) = L$, $c \in L$, und durch Produktbildung folgt: $G$ enthält alle Transvektionen mit Vektor $Eb$, wo $b = \sum a_ic_i$ ein beliebiges Element aus dem Ideal $J = o(\tau)$ ist.

Sei schliesslich $\tau'$ eine beliebige Transvektion der Ordnung $o(\tau') = J' \subset o(\tau) = J$.

$$\tau' X = X + A'\phi'(X) \tag{11}$$

sei eine Darstellung von $\tau'$. Wir schreiben $A' = \sum E'_i a'_i$, $1 \leqq i \leqq n-1$, wobei die $E'_i$, $1 \leqq i \leqq n-1$, eine Basis von $H' = \phi'^{-1}(0)$ bilden. Wegen $a'_i \in J$ gehören die Transvektionen

$$\tau'_i X = X + E'_i a'_i \phi'(X) \tag{12}$$

zu $G$, also auch $\tau' = \prod \tau'_i \in G$.

2. 6. 2. Wir betrachen jetzt den Fall $n \geqq 3$ mit einer beliebigen Menge $(\tau_\alpha)_{\alpha \in A}$. Sei $J$ das von den $J_\alpha = o(\tau_\alpha)$ erzeugte Ideal. Eine beliebige Transvektion $\tau'$ mit $o(\tau') = J' \subset J$ besitzt eine Darstellung (11) mit $A' = \sum E'_i a'_i$, $1 \leqq i \leqq n-1$, $a'_i \in J' \subset J$. Die Überlegungen aus 2. 6. 1 zeigen, dass $G$ die Transvektionen $\tau'_i$, $1 \leqq i \leqq n-1$, der Form (12) enthält, also auch $\tau' = \prod \tau'_i \in G$.

2. 6. 3. Wir betrachten den Fall $n = 2$ und nehmen zunächst wiederum an, dass die Menge $(\tau_\alpha)_{\alpha \in A}$ nur aus dem Element $\tau$ besteht. Durch geeignete Wahl der Basis lässt such $\tau$ in der Form

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \tag{13}$$

darstellen, wo $(u) = o(\tau)$ ist. Dann repräsentiert auch

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} 1 & ua^2 \\ 0 & 1 \end{pmatrix}, \qquad a \in L^*, \tag{14}$$

ein Element aus der von $\tau$ erzeugten, unter $SL(2, L)$ invarianten Gruppe $G$. Mit der Matrix (13) enthält $G$ also auch die Matrizen

$$\begin{pmatrix} 1 & u(\sum \pm a_i^2) \\ 0 & 1 \end{pmatrix}, \qquad a_i \in L^*. \tag{15}$$

Da $2 \in L^*$, haben wir für jedes $c \in L$ die Darstellung

$$c = (c+1)^2/2^2 - (c-1)^2/2^2. \tag{16}$$

Falls also $c+1 \in L^*$ und $c-1 \in L^*$, so haben wir aus (15) und (16), dass $G$ auch die Matrix

$$\begin{pmatrix} 1 & uc \\ 0 & 1 \end{pmatrix} \tag{17}$$

enthält. Falls dagegen $c+1 \in I$ oder $c-1 \in I$, so ersetzen wir $c$ zunächst durch $c' = c+1$ beziehungsweise durch $c' = c-1$. Dann ist $c'+1 \in L^*$ und $c'-1 \in L^*$ und auf Grund von (15) und (16) enthält $G$ also eine Matrix (17), in der $c$ durch $c'$ ersetzt ist. Da auch (13) zu $G$ gehört, folgt, dass auch (17) zu $G$ gehört.

Da ein 1-dimensionaler Unterraum von $V = V_2(L)$ in einen anderen 1-dimensionalen Unterraum von $V$ stets durch ein Element $\sigma \in SL(2, L)$ übergeführt werden kann, folgt, dass $G$ mit der Transvektion $\tau$, (13), der Ordnung $o(\tau) = (u)$ auch alle Transvektionen $\tau'$ der Ordnung $o(\tau') \subset o(\tau) = (u)$ enthält.

Wir betrachten jetzt eine beliebige Menge $(\tau_\alpha)_{\alpha \in A}$ von Transvektionen $\tau_\alpha$. Sei $J$ das von den $J_\alpha = o(\tau_\alpha) = (u_\alpha)$ erzeugte Ideal. Eine beliebige Transvektion $\tau'$ der Ordnung $o(\tau') = (u') \subset J$ lässt sich darstellen durch (13) mit $u'$ statt $u$. $u'$ lässt sich schreiben als $u' = \sum u_\alpha c_\alpha$, $c_\alpha = 0$ für fast alle $\alpha \in A$. Die von den $\tau_\alpha$ erzeugte, unter $SL(2, L)$ invariante Untergruppe $G$ von $GL(2, L)$ enthält, wie wir soeben sahen, alle Transvektionen (17) mit $u_\alpha c_\alpha$ an Stelle von $uc$. Also gehört auch das Produkt dieser Transvektionen zu $G$, d.h. $\tau' \in G$.

2.7. *Beweis von Satz 3 für $n = 2$.*

2.7.1. Angenommen, $G$ enthält das Element $\rho$ der Form $\begin{pmatrix} a & u \\ 0 & a+v \end{pmatrix}$ mit $(v) = K$. $\left(\begin{matrix} 1 & -a-v \\ 0 & 1 \end{matrix}\right.$ Transvektion $\tau$. Dann gehört auch $\rho\tau\rho^{-1}\tau^{-1}$: $\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}$ zu $G$, das heisst, $G$ enthält eine Transvektion der Ordnung $(v) = K$ und daher, nach Satz 2, $SC(n, L, K) \subset G$.

2.7.2. Wir betrachten den Fall, das $o(G) = J \subset I$. Ein Element $\sigma \in G$ besitzt die Darstellung

$$\begin{pmatrix} b & x \\ y & b+z \end{pmatrix} \tag{18}$$

mit $x, y, z \in J = o(G)$, $b \in L^*$. Es ist $o(\sigma) = (x, y, z) =$ das von $x$, $y$ und $z$ erzeugte Ideal in $L$. Wenn wir zeigen, dass $G$ mit $\sigma$, (18), auch die Transvektionen

$$\begin{pmatrix} 1 & y-x \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & y-2x\mp z \\ 0 & 1 \end{pmatrix} \tag{19}$$

enthält, deren Ordnungen offenbar gerade das Ideal $o(\sigma)$ erzeugen, dann haben wir auf Grund von Satz 2, dass $SC(2, L, o(\sigma)) \subset G$. Da dies für jedes $\sigma \in G$ gilt, folgt $SC(2, L, J) \subset G$.

Wir beweisen die Existenz der Transvektionen (19) in zwei Schritten.

2.7.2.1. Da $G$ invariant ist unter $SL(2, L)$, enthält $G$ mit $\sigma$, (18), auch das Element

$$(20) \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} b & x \\ y & b+z \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} b & x \\ y & b+z \end{pmatrix}$$
$$= \begin{pmatrix} b^2+bz-y^2 & bx+xz-by-yz \\ -bx+by & b^2+bz-x^2 \end{pmatrix} \underset{(\text{kurz})}{=} \begin{pmatrix} b' & x' \\ y' & b'+z' \end{pmatrix}.$$

Dann gehört auch $\sigma''$:

$$(21) \quad \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} b' & x' \\ y' & b'+z' \end{pmatrix}\begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} b'+ky' & -k^2y'+x'+kz' \\ y' & b'-ky'+z' \end{pmatrix}$$

zu $G$. Wenn wir in (21) setzen: $k = (x+y)/2b$, so erhalten wir für $\sigma'' \in G$ die Darstellung

$$(22) \quad \begin{pmatrix} b^2+bz-(y^2+x^2)/2 & (y^2-x^2)(y+x)/4b-(b+z)(y-x) \\ b(y-x) & b^2+bz-(y^2+x^2)/2 \end{pmatrix}$$
$$= (\text{kurz}) \begin{pmatrix} a & u \\ v & a \end{pmatrix}.$$

Das Element $\tau \in SL(2, L)$ sei erklärt durch $\begin{pmatrix} 1+v & -a(1+v)^{-1}(2+v) \\ 0 & (1+v)^{-1} \end{pmatrix}$. Dann gehört auch $\rho' = \tau^{-1}\sigma''^{-1}\tau\sigma''$ zu $G$. Wir finden für $\rho'$:

$$(23) \quad (a^2-uv)^{-1}\begin{pmatrix} a^2-a^2v(2+v)-uv(1+v)^{-2} & \cdots \\ 0 & a^2-a^2v(2+v)-uv(1+v)^2 \end{pmatrix}$$
$$= (\text{kurz}) \begin{pmatrix} a' & u' \\ 0 & a'+v' \end{pmatrix} \text{mit } (v') = (v) = (y-x).$$

Aus 2.7.1, angewandt auf das Element $\rho'$, (23), folgt dass $G$ die erste der Transvektionen (19) enthält.

2.7.2.2. Mit dem Element $\sigma$, (18), enthält $G$ auch die Elemente

$$(24) \quad \begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix}\begin{pmatrix} b & x \\ y & b+z \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \mp 1 & 1 \end{pmatrix} = \begin{pmatrix} b \mp x & x \\ y-x\mp z & b\pm x+z \end{pmatrix}.$$

Indem wir auf die Elemente (24) dieselben Überlegungen anwenden, die wir auf das Element $\sigma$, (18), in 2.7.2.1 angewandt haben, erkennen wir, dass $G$ auch die Transvektionen der Ordnung $(y-2x\mp z)$ enthält, also die letzten beiden der Transvektionen (19).

2.7.3. Wir betrachten jetzt den Fall dass $o(G) = J = L$. Dann enthält $G$ ein Element $\sigma$ der Form

(25) $$\begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix}, \qquad a \in L^*.$$

Mit dem Element $\tau$: $\begin{pmatrix} bd & d^{-1} \\ -d & 0 \end{pmatrix}$ aus $SL(2,L)$ gehört dann auch $\rho = \tau^{-1}\sigma^{-1}\tau\sigma$:

(26) $$\begin{pmatrix} -a^{-1}d^{-2} & -b-a^{-1}d^{-2}b \\ 0 & -ad^2 \end{pmatrix}$$

zu der Gruppe $G$. Um 2.7.1 anwenden zu können, müssen wir $d \in L^*$ so wählen, dass $a^{-1}d^{-2} - ad^2 \in L^*$, d.h.

(27) $$a^2d^4 - 1 \in L^*.$$

Da wir $\mathrm{char}(L/I) \neq 2$ und $L/I \neq \boldsymbol{F}_3$ vorausgesetzt haben, existiert ein $d \in L^*$ mit (27) stets dann, wenn $L/I \neq \boldsymbol{F}_5$ est, denn dann hat $L/I$ wenigstens 6 Elemente. Wenn (27) gilt, dann folgt aus 2.7.1 mit $\rho$, (26), dass $G$ eine Transvektion der Ordnung $L$ enthält, also nach Satz 2, $SL(2,L) \subset G$.

2.7.3.1. Es bleibt also der Fall $L/I = \boldsymbol{F}_5$ zu betrachten. Falls $g_I a^2 \neq 1$ ist, dann gibt es ein $d \in L^*$ so, dass (27) gilt. Wir können uns also auf den Fall $g_I a = \pm 1$ beschränken.

Zunächst können wir dann $d \in L^*$ so wählen, dass $g_I a d^2 = 1$ ist. Neben dem Element $\rho$, (26), enthält $G$ auch das Element $\rho'$, das aus (26) entsteht, wenn man $d$ ersetzt durch $a^{-1}d^{-1}$. Dann gehört auch $\rho\rho'$ zu $G$. $\rho\rho'$ hat die Form

(28) $$\begin{pmatrix} 1 & b(1+a^{-1}d^{-2})^2 \\ 0 & 1 \end{pmatrix}.$$

$\rho\rho'$ ist also eine Transvektion der Ordnung $(b)$. Falls $(b) = L$, so haben wir mit Satz 2 $SL(2,L,L) = SL(2,L) \subset G$. Falls $(b) \subset I$, so haben wir mit Satz 2 $SL(2,L,(b)) \subset G$. Dann gehört auch das Element

(29) $$\begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix}\begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}$$

zu $G$.

Wir haben also jetzt ein Element $\sigma$, (25), mit $b = 0$ und $g_I a = \pm 1$ in $G$. Damit enthät $G$ auch das Element $\rho$, (26), mit $d = 1$. Es ist also $o(\rho) = J' = (a^2 - 1) \subset I$. Nach 2.7.2 ist dann jedenfalls $SC(2,L,J') \subset G$.

Wir betrachten die Fälle: (i) $J' = (a-1)$, (ii) $J' = (a+1)$. Im Falle (i) wählen wir für das Element $\sigma$, (29), eine neue Basis so, dass $\sigma$ die Darstellung erhält:

$$(30)\qquad \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}0 & a\\ 1 & 0\end{pmatrix}\begin{pmatrix}\frac12 & \frac12\\ \frac12 & -\frac12\end{pmatrix}=\begin{pmatrix}(1+a)/2 & (1-a)/2\\ (-1+a)/2 & (-1-a)/2\end{pmatrix}.$$

Es ist also $h_{J'}\sigma$: $\begin{pmatrix}1 & 0\\ 0 & 1-2\end{pmatrix}$. Aus 2.7.1 folgt, dass $SL(2,L/J')\subset h_{J'}G$.

Im Falle (ii) wählen wir für $\sigma$, (29), eine Basis wie folgt:

$$(31)\qquad \begin{pmatrix}1 & 2\\ 1 & -2\end{pmatrix}\begin{pmatrix}0 & a\\ 1 & 0\end{pmatrix}\begin{pmatrix}\frac12 & \frac12\\ \frac14 & -\frac14\end{pmatrix}=\begin{pmatrix}(4+a)/4 & (4-a)/4\\ (-4+a)/4 & (-4-a)/4\end{pmatrix}.$$

Es ist also $h_{J'}\sigma$: $\begin{pmatrix}2 & 0\\ 0 & 2-4\end{pmatrix}$. Aus 2.7.1 folgt wiederum $SL(2,L/J')\subset h_{J'}G$. Da $\mathrm{Kern}(h_{J'}\mid_{SL(n,L)})=SC(n,L,J')\subset G$, folgt $SL(2,L)\subset G$.

2.8. *Beweis von Satz 3 für* $n\geqq 3$.

2.8.1. Angenommen, $G$ enthält ein Element $\rho$ so, dass $\rho X - X\in H$ für alle $X\in V$, wo $H$ ein Unterraum der Kodimension 1 ist. Behauptung: Dann enthält $G$ auch die Transvektion $\tau$ der Form

$$(32)\qquad \tau X = X + (\rho U - U)\phi(X)$$

wo $\phi^{-1}(0)=H$ und $U$ ein beliebiger Vektor $H$ ist. In der Tat, setze $\tau' X = X + U\phi(X)$. Dann gehört auch $\tau=\rho\tau'\rho^{-1}\tau'^{-1}$ zu $G$. Wegen $\phi(\rho X) = \phi(X)$ und $\phi(\rho^{-1}U)=\phi(\rho U)=0$ hat $\tau$ die Gestalt (32).

2.8.2. Sei $\sigma\in G$. Das Ideal $o(\sigma)\subset J=o(G)$ wird erzeugt von Elementen $a\in L$, deren jedes in folgender Weise beschrieben werden kann: Es gibt einen Vektor $E$ der Ordnung $o(E)=L$ und eine Linearform $\phi$ der Ordnung $L$ so, dass $\phi(E)=0$ und $\phi(\sigma E)=a$. Wenn wir zeigen, wass, für jedes solche $a$, $G$ eine Transvektion der Ordnung $(a)$ enthält, dann ist der Satz 3 bewiesen.

Denn jedenfalls enthält $G$ dann, auf Grund von Satz 2, die Gruppen $SC(n,L,o(\sigma))$ für jedes $\sigma\in G$, und da $J=o(G)$ erzeugt wird von $o(\sigma)$, $\sigma\in G$, folgt die Behauptung mit Satz 2.

2.8.3. Sei $\sigma\in G(n,L)$. Sei $E$ ein Vektor der Ordnung $L$ und $\phi$ eine Linearform der Ordnung $L$ mit $\phi(E)=0$ und sei $\phi(\sigma E)=a$. *Behauptung*: Es gibt ein $\sigma'$ in der von $\sigma$ erzeugten, unter $SL(n,L)$ invarianten Untergruppe $G$ von $GL(n,L)$ und es gibt eine Basis $E'_i$, $1\leqq i\leqq n$, mit dualer Basis $\phi'_i$, $1\leqq i\leqq n$, so, dass $\phi'_n(\sigma' E_i)=0$ für alle $i$ mit $1\leqq i\leqq n-2$ und so, dass $\phi'_k(\sigma' E'_1)=a$ für ein $k$ mit $1\leqq k\leqq n-1$.

2.8.3.1. Zum Beweis dieser Behauptung gehen wir aus von einer Basis

$E_i$, $1 \leqq i \leqq n$, und dualer Basis $\phi_i$, $1 \leqq i \leqq n$, so, dass $E = E_1$ und $\phi = \phi_{n-1}$. Die Matrix von $\sigma$ und die Matrix von $\sigma^{-1}$ bezüglich dieser Basis sei bezeichnet durch

$$((a_{ik})) = ((\phi_i(\sigma E_k))), \qquad ((b_{ik})) = ((\phi_i(\sigma^{-1} E_k))).$$

Insbesondere ist also $a = \phi_{n-1}(\sigma E_1) = a_{n-1\,1}$.

Wir definieren die Transvektionen $\tau_j$, $j \neq n-1$, durch

$$(33) \qquad \tau_j X = X + E_j \phi_{n-1}(X) \qquad (j \neq n-1).$$

Da $G$ invariant ist, gehört auch $\rho_j = \sigma^{-1}\tau_j \sigma \tau_j^{-1}$ zu $G$. Wir finden

$$(34) \qquad \rho_j E_k = E_k + \sigma^{-1} E_j \phi_{n-1}(\sigma E_k) \qquad (j, k \neq n-1).$$

Ferner, für $j \neq n-1$,

$$(35) \qquad \rho_j \sigma^{-1} E_j = \sigma^{-1}\tau_j \sigma \tau_j^{-1} \sigma^{-1} E_j = \sigma^{-1} E_j (1 - a_{n-1\,j} b_{n-1\,j}) - E_j b_{n-1\,j}.$$

2.8.3.2. Wir betrachten zunächst den Fall: Es gibt ein $r$, $2 \leqq r \leqq n$, mit $b_{rn} \in L^*$. In diesem Falle erklären wir eine neue Basis $E'_i$ durch: $E'_i = E_i$ für $i \neq r$ und $E'_r = \sigma^{-1} E_n$.

Für $i < n-1$ und $i \neq r$ haben wir wegen (34)

$$(36) \qquad \rho_n E'_i = \rho_n E_i = E_i + \sigma^{-1} E_n \phi_{n-1}(\sigma E_i) = E'_i + E'_r a_{n-1\,i}$$

und für $r \neq n-1$ haben wir wegen (35)

$$(37) \qquad \rho_n E'_r = \rho_n \sigma^{-1} E_n = E'_r (1 - a_{n-1\,n} b_{n-1\,n}) - E'_n b_{n-1\,n}.$$

*Fall* (i): $r \neq n-1$. Aus (36) und (37) folgt:

$$\phi'_{n-1}(\rho_n E'_i) = 0 \text{ für } i < n-1; \; \phi'_r(\rho_n E'_i) = a_{n-1\,1}$$

Wenn wir also $E'_{n-1}$ und $E'_n$ vertauschen und $\sigma' = \rho_n$ setzen, so folgt, mit $k = r$, die Behauptung 2.8.3.

*Fall* (ii): $r = n-1$. Aus (36) und (37) folgt

$$\phi'_n(\rho_n E'_i) = 0 \text{ für } i < n-1; \; \phi'_{n-1}(\rho_n E'_1) = a_{n-1\,1}.$$

Mit $\sigma' = \rho_n$, $k = n-1$ folgt die Behauptung 2.8.3.

2.8.3.3. Wir betrachten den in 2.8.3.2 ausgeschlossenen Fall, dass $b_{rn} \in I$ für alle $r$, $2 \leqq r \leqq n$. Dann gibt es jedoch ein $r$, $2 \leqq r \leqq n$, so, dass $b_{r1} \in L^*$, wie man durch Reduktion $\bmod I$ erkennt. Wir erklären eine neue Basis $E'_i$ durch $E'_i = E_i$ für $i \neq r$, und $E'_r = \sigma^{-1} E_1$. Aus (34) und (35) folgt

(38) $$\rho_1 E'_i = E'_i + E'_r a_{n-1\,i} \text{ für } i \neq n-1,\ i \neq r,$$

(39) $$\rho_1 E'_r = E'_r(1 - a_{n-1\,1} b_{n-1\,1}) - E'_1 b_{n-1\,1}.$$

*Fall* (i) : $r = n$. Aus (38) und (39) folgt

$$\phi'_{n-1}(\rho_1 E'_i) = 0 \text{ für } i < n-1;\ \phi'_n(\rho_1 E'_1) = a_{n-1\,1}$$

Wenn wir $E'_{n-1}$ und $E'_n$ vertauschen und $\sigma' = \rho_1$ setzen, so folgt, mit $k = r$, die Behauptung 2. 8. 3.

*Fall* (ii) : $r \neq n$. Aus (38) und (39) folgt

$$\phi'_n(\rho_1 E'_i) = 0 \text{ für } i < n-1;\ \phi'_r(\rho_1 E'_1) = a_{n-1\,1}.$$

Mit $\sigma' = \rho_1$, $k = r$ folgt die Behauptung 2. 8. 3.

2. 8. 4. Wir kommen jetzt zum Beweis der in 2. 8. 2 aufgestellten Behauptung, dass die Gruppe $G$ eine Transvektion der Ordnung ($a$) enthält. Nach 2. 8. 3 können wir annehmen, dass wir eine Basis $E_i$, $1 \leqq i \leqq n$, von $V$ haben mit dualer Basis $\phi_i$, $1 \leqq i \leqq n$, und ein Element $\sigma \in G$ so, dass

(40) $$\phi_n(\sigma E_i) = 0 \text{ für } i < n-1;\ \phi_k(\sigma E_1) = a \text{ für ein } k,\ 2 \leqq k \leqq n-1.$$

Wir betrachten die Transvektion $\tau$:

(41) $$\tau X = X + E_1 \phi_n(X).$$

Für $\rho = \sigma\tau\sigma^{-1}\tau^{-1} \in G$ finden wir

(42) $$\rho X = X - E_1\phi_n(X) + \sigma E_1 \phi_n(\sigma^{-1}X) - \sigma E_1 \phi_n(\sigma^{-1}E_1)\phi_n(X).$$

Es ist also $\rho X - X \in H$, wo $H$ die von $E_1, \cdots, E_{n-1}$ aufgespannte Hyperbene ist. Daher folgt aus 2. 8. 1, mit $U = E_i$, $1 \leqq i \leqq n-1$: $G$ enthält die Transvektionen mit dem Vektor $\rho E_i - E_i = \sigma E_1 \phi_n(\sigma^{-1}E_i)$, $1 \leqq i \leqq n-1$. Wir unterscheiden zwei Fälle:

2. 8. 4. 1. Eines der Elemente $\phi_n(\sigma^{-1}E_i)$, $1 \leqq i \leqq n-1$, gehört zu $L^*$. Dann enthält $G$ also eine Transvektion der Ordnung $L$ und mit Satz 2 folgt $SL(n, L) \subset G$, $o(G) = J = L$, es folgt also die Behauptung von Satz 3.

2. 8. 4. 2. Die Elemente $\phi_n(\sigma^{-1}E_i)$, $1 \leqq i \leqq n-1$, gehören alle zu $I$. Wir bezeichnen mit $K$ das von diesen Elementen erzeugte Ideal. Dann ist also $K \subset I$. $K$ stimmt überein mit dem von den Elementen $\phi_n(\sigma E_i)$, $1 \leqq i \leqq n-1$, erzeugten Ideal; denn $h_K \sigma^{-1} g_K H = g_K H$ impliziert $h_K \sigma g_K H = g_K H$ und umgekehrt. Wegen Satz 2 und dem Schluss von 2. 8. 4 haben wir $SC(n, L, K) \subset G$.

Wir definieren eine Linearform $\psi$ durch

(43) $$\psi(\sigma E_i)=0 \text{ für } i<n-1;\ \psi(E_n)=0;\ \psi(\sigma E_{n-1})=1.$$

Da die $g_K E_i$, $1\leqq i\leqq n-1$, durch $h_K\sigma$ unter sich transformiert werden und $K\subset I$, bilden die $\sigma E_i$, $1\leqq i\leqq n-1$, zusammen mit $E_n$, eine Basis für $V$; daher ist $\psi$ durch (43) wohldefiniert.

Wir betrachten die Transvektion $\mu$:

(44) $$\mu X=X+E_n\phi_n(\sigma E_{n-1})\psi(X).$$

Offenbar ist $o(\mu)\subset K$, also $\mu\in G$. Dann gilt auch $\sigma'=\mu^{-1}\sigma\in G$. Wir finden

$$\sigma' E_i=E_i,\ i<n-1;\ \sigma' E_{n-1}=\sigma E_{n-1}-E_n\phi_n(\sigma E_{n-1}).$$

Es ist also $\sigma' H=H$, wo $H=\phi_n^{-1}(0)$. Folglich $\phi_n\sigma'^{-1}=d\phi_n$, mit $d\in L^*$.

Mit der Tranvektion $\tau$, (41), erhalten wir für $\rho'=\sigma'\tau\sigma'^{-1}\tau^{-1}\in G$ nach (42) den Ausdruck

(45) $$\rho' X=X+(\sigma E_1 d-E_1)\phi_n(X).$$

$\rho'$, (45), ist also eine Transvektion der Ordnung $o(\sigma E_1 d-E_1)\supset(\phi_k(\sigma E_1))=(a)$. Nach Satz 2 enthält $G$ also eine Transvektion der Ordnung $(a)$. Damit ist die Behauptung in 2.8.2 bewiesen, und also Satz 3 bewiesen.

2.9. Das Theorem 3 ist eine Zusammenfassung der vorstehend bewiesenen Ergebnisse.

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# ON DIFFERENTIAL EQUATIONS AND THE FUNCTION $J_\mu^2 + Y_\mu^2$.*[1]

By PHILIP HARTMAN.

**Introduction.** Let $J_\mu = J_\mu(t)$, $Y_\mu = Y_\mu(t)$ denote Bessel functions of a non-negative order $\mu$ of the first and second kind, respectively, and let $t$ be real. It is known that $t(J_\mu^2 + Y_\mu^2)$ is increasing or decreasing for $t > 0$ according as $\mu < \frac{1}{2}$ or $\mu > \frac{1}{2}$ and that $(t^2 - \mu^2)^{\frac{1}{2}}(J_\mu^2 + Y_\mu^2)$ is increasing for $t \geqq \mu \geqq 0$; cf. [9], p. 446. These facts are usually derived from Nicholson's integral formula for $J\mu^2 + Y_\mu^2$, the proof of which is rather involved. It seems, therefore, of interest to obtain these assertions directly from simple, general theorems on differential equations.

Let $q = q(t)$ be a continuous, positive function for large $t$. General conditions (cf. [11]) are known which imply that

$$(0.1) \qquad u'' + q(t)u = 0$$

has a pair of real-valued solutions $u = x(t), y(t)$ satisfying, as $t \to \infty$, asymptotic formulae similar to

$$(0.2) \qquad q^{\frac{1}{4}}z = \exp i \int^t q^{\frac{1}{2}}(s)ds + o(1),$$

$$(q^{\frac{1}{4}}z)' = iq^{\frac{1}{2}} \exp i \int^t q^{\frac{1}{2}}(s)ds + o(1),$$

where $z = x + iy$. But these results give no information on the possible monotone character of $q^{\frac{1}{4}} | z |$ or on sharp bounds for $q^{\frac{1}{4}} | z |$ for all $t$-values under consideration. The object of Part I of this paper is to supply such information. It will be shown there that, under suitable conditions on $q$, the number 1 is a bound for $q | z |^4$ and that this implies the monotony of $| z |$. The question of the monotony of $q | z |^4$ can be decided in many cases by examining the applicability of this result to the differential equation having $q^{\frac{1}{4}}x$, $q^{\frac{1}{4}}y$ as solutions; cf. the change of variables (2.7) below.

These results will be applied in Section 4 to the Bessel functions. It

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will be seen that they imply, in particular, that $z = t^{\frac{1}{2}}(J_\mu + iY_\mu)$ satisfies, for $t > 0$,

$$|z| > 0, \qquad |z|' \leqq 0, \qquad |z|'' \geqq 0 \tag{0.3}$$

or

$$|z| > 0, \qquad |z|' \geqq 0, \qquad |z|'' \leqq 0 \tag{0.4}$$

according as $\mu > \frac{1}{2}$ or $\mu \leqq \frac{1}{2}$. Note that the last part of (0.3), $|z|'' \geqq 0$, is not implied by the following consequence

$$|z|^2 > 0, \ (|z|^2)' \leqq 0, \ (|z|^2)'' \leqq 0, \ (|z|^2)''' \geqq 0, \cdots \tag{0.5}$$

of Nicholson's formula (for $\mu \geqq \frac{1}{2}$).

On the other hand, the results and methods of Part I do not lead to the sequence of inequalities in (0.5). The question of higher order monotony of $|z(t)|^2$, where $z = x(t) + iy(t)$ is a complex-valued solution of a general equation (0.1), will be examined in Part IV. The results to be obtained will depend on Parts II and III which deal with differential equations of arbitrary order. The methods to be employed will be very different from and do not depend on those in Part I.

It will remain undecided whether or not the results on the higher order monotony of $|z|^2$ can be sharpened to give such results about $|z|$.

Applications of the results of Part IV to Bessel functions give the complete monotony (i.e., (0.5)) for $|z|^2 = t(J_\mu{}^2 + Y_\mu{}^2)$ for $t > 0$ when $\mu > \frac{1}{2}$. This proof is longer than the one involving Nicholson's formula but has the advantage of applying to a large class of differential equations.

It can be mentioned that Nicholson's formula implies that, when $\mu < \frac{1}{2}$, the derivative of $|z|^2 = t(J_\mu{}^2 + Y_\mu{}^2)$ is completely monotone for $t > 0$. It will remain undecided how to derive this result from a general theory of differential equations. A partial result in this direction is given in Section 22.

## Part I. Monotony and convexity of $|z|$.

This part of the paper concerns inequalities of the form (0.3) or (0.4) for some complex-valued solution $u = z(t)$ of (0.1). The main result is Theorem 3.1.

**1. Bounds for solutions.** Let $Q = Q(s)$ be a continuous, monotone function of $s$ for $s > S$ satisfying, as $s \rightarrow \infty$,

$$Q \rightarrow 1. \tag{1.1}$$

Then

$$d^2U/ds^2 + Q(s)U = 0 \tag{1.2}$$

has a pair of real-valued solutions $U = X(s), Y(s)$ such that $Z = X + iY$ satisfies, as $s \to \infty$,

$$Z = \exp i \int^s Q^{\frac{1}{2}}(\tau)\,d\tau + o(1), \qquad Z_s = iZ + o(1), \tag{1.3}$$

where $Z_s = dZ/ds$; [12], Appendix. The solution $U = Z(s)$ of (1.2) satisfying (1.3) is uniquely determined, up to constant factors, by the requirement that

$$Z_s/Z \to i \quad \text{as} \quad s \to \infty; \tag{1.4}$$

[6]. In view of (1.3), the (constant) Wronskian of $X$ and $Y$ has the value 1,

$$XY_s - X_sY = 1. \tag{1.5}$$

LEMMA 1.1. *Let $Q = Q(s)$ be positive, continuous and monotone for $S < s < \infty$ and satisfy* (1.1). *Let $U = Z(s)$ be the solution of* (1.2) *satisfying* (1.3) *as $s \to \infty$. Then, for $s > S$,*

$$|Z|^2Q \leqq 1 \leqq |Z|^2 \text{ and } |Z_s|^2 \leqq 1 \leqq |Z_s|^2/Q \tag{1.6$_1$}$$

*or*

$$|Z|^2Q \geqq 1 \geqq |Z|^2 \text{ and } |Z_s|^2 \geqq 1 \geqq |Z_s|^2/Q \tag{1.6$_2$}$$

*according as*

$$(1.7_1) \quad dQ \geqq 0 \qquad or \qquad (1.7_2) \quad dQ \leqq 0.$$

It will be clear from the proof that the interval $S < s < \infty$ can be replaced by an interval $S < s < S^*(\leqq \infty)$, if the assumptions (1.1), (1.3) refer to $s \to S^*$. If $S^* < \infty$, then $Q$ can be defined at $s = S^*$ by $Q(S^*) = 1$ and is then continuous for $S < s \leqq S^*$. The existence of a solution $U = Z(s)$ satisfying (1.3) is obvious.

*Remark* 1. In addition to (1.6), it can be shown that

$$|Z|^2Q + |Z_s|^2 \leqq 1 + Q \tag{1.8}$$

holds in both cases $(1.7_1)$ and $(1.7_2)$. It will be clear that strict inequality holds in (1.6) and (1.8) if $Q(s) \not\equiv 1$ for large $s$.

*Remark* 2. The inequality $|Z|^2 \geqq 1$ in $(1.6_1)$ holds for $s > S$ if the conditions $Q > 0$, $dQ \geqq 0$ for $s > S$ are replaced by $Q(s) \leqq 0$ for $S < s \leqq S_0$

and $Q(s) > 0$, $dQ(s) \geqq 0$ for $s > S_0$, where $S_0$ is some fixed number. Note that $Q$ is not required to be monotone on the interval $S < s \leqq S_0$.

*Proof of Lemma* 1.1. If (1.2) is divided by $Q(s)$ and differentiated with respect to $s$, and the result divided by $Q(s)$, one obtains the equation

$$d^2W/d\tau + Q^{-1}(s)W = 0, \text{ where } s = s(\tau),$$

and

$$W = U', \; d\tau = Q(s)\,ds, \text{ and } dW/d\tau = U''/Q = -U.$$

This makes it clear that it is sufficient to consider only the case $(1.6_1)$-$(1.7_1)$. (This change of variables, which will be used several times in this paper, is suggested in part by Wintner [13]; cf. [3].)

Let $U = U(s)$ be any real-valued solution of (1.2). Then, by virtue of (1.2) and $(1.7_1)$,

$$(1.9) \quad d(U^2Q + U_s^2) = U^2dQ \geqq 0 \text{ and } d(U^2 + U_s^2/Q) = -(U_s/Q)^2dQ \leqq 0.$$

For arbitrarily fixed real $\phi$, let

$$(1.10) \qquad U = X(s)\cos\phi + Y(s)\sin\phi;$$

so that (1.1) and (1.3) imply $U^2Q + U_s^2 \rightarrow 1$ and $U^2 + U_s^2/Q \rightarrow 1$ as $s \rightarrow \infty$. Hence, by (1.9), for $s > S$,

$$(1.11) \qquad U^2Q + U_s^2 \leqq 1 \text{ and } U^2 + U_s^2/Q \geqq 1.$$

The first inequality in (1.11) shows that $U^2Q \leqq 1$ for $s > S$ and for every $\phi$. For a fixed $s$, choose $\phi$ in (1.10) so that $U(s) = (X^2(s) + Y^2(s))^{\frac{1}{2}} = |Z|$. Then, for this value of $s$, it is seen that $|Z|^2Q = U^2Q \leqq 1$.

The second inequality in (1.11) implies that $U^2(s) \geqq 1$ if $U_s(s) = 0$. For a fixed $s$, choose $\phi$ so that $U_s(s) = 0$. By Cauchy's inequality, $U^2(s) \leqq |Z(s)|^2$. Thus, for this value of $s$, $|Z(s)|^2 \geqq U^2(s) \geqq 1$.

This proves the first two inequalities in $(1.6_1)$. The last two are proved similarly. Hence Lemma 1.1 follows.

*On Remark* 1. Again, only the case $(1.6_1)$-$(1.7_1)$ will be considered. In view of (1.10), the expression $U^2Q + U_s^2$ is a quadratic form in $(\cos\phi, \sin\phi)$ for a fixed $s$. According to (1.11), the eigenvalues $\lambda$ of this form satisfy $\lambda \leqq 1$. It is readily verified, if use is made of (1.5), that the eigenvalues $\lambda$ of this form are the roots of the equation $\lambda^2 - \lambda(|Z|^2Q + |Z_s|^2) + Q = 0$. The inequality (1.8) merely expresses the fact that the largest root of this equation is at most 1.

*On Remark* 2. In view of Lemma 1.1, it follows that $|Z(s)| \geqq 1$, $|Z_s(s)| \leqq 1$ for $s \geqq S_0$. Choose $\phi$ so that $\cos\phi = Y_s(S_0)/|Z_s(S\ )|$, $\sin\phi = -X_s(S_0)/|Z_s(S_0)|$. Then $U(s)$ in (1.10) satisfies

$$U(S_0) = (XY_s - X_sY)/|Z_s| = 1/|Z_s(S_0)| \geqq 1, \quad U_s(S_0) = 0.$$

Initial conditions $U(S_0) > 0$, $U_s(S_0) = 0$ at $s = S_0$ and $Q \leqq 0$ for $S < s \leqq S_0$ imply, by a simple convexity argument, that $dU(s) \leqq 0$ on this interval. Hence $U(s) \geqq U(S_0) = 1$ for $S < s \leqq S_0$. Cauchy's inequality gives therefore $|Z(s)| \geqq U(s) \geqq 1$.

**2. Bounds for $q\,|z|^4$.** The desired inequalities for solutions of (0.1) do not follow directly from Lemma 1.1 but can be obtained from this lemma after a standard change of variables (Riemann-Liouville).

LEMMA 2.1. *Let $q(t)$ be a positive function of class $C^2$ for $t > T_0$ with the properties that*

$$(2.1) \qquad Q \equiv 1 + 5q'/16q^3 - q''/4q^2 \equiv 1 - (q'/q^{\frac{5}{4}})'/4q^{\frac{3}{4}}$$

*satisfies $Q(t) \to 1$ as $t \to \infty$ and that either*

$$(2.2_1) \qquad dQ \leqq 0 \text{ for } t > T_0$$

*or there exists a $T_1 \geqq T_0$ such that*

$$(2.2_2) \qquad Q \leqq 0 \text{ for } T_0 < t \leqq T_1 \text{ and } Q > 0,\ dQ \geqq 0 \text{ for } t > T_1.$$

*Then (0.1) possesses a pair of real-valued solutions $u = x(t), y(t)$ such that $z = x + iy$ satisfies, as $t \to \infty$,*

$$(2.3) \qquad q^{\frac{1}{4}}z \sim \exp i \int^t Q^{\frac{1}{2}}(\tau)\, q^{\frac{1}{2}}(\tau)\, d\tau, \qquad (q^{\frac{1}{4}}z)' \sim i q^{\frac{1}{2}}(q^{\frac{1}{4}}z);$$

$$(2.4) \qquad xy' - x'y = 1;$$

*and, for $t > T_0$, either*

$$(2.5_1) \qquad q\,|z|^4 \leqq 1 \qquad \text{or} \qquad (2.5_2) \qquad q\,|z|^4 \geqq 1$$

*according as $(2.2_1)$ or $(2.2_2)$ holds.*

If $q$ is of class $C^3$, then $Q(t)$ has a derivative given by

$$Q' = (18qq'q'' - 15q'^3 - 4q^2q''')/16q^4.$$

Note that $dQ \geqq 0$ is implied by

$$(2.6) \qquad q > 0,\ q' \geqq 0,\ q'' \leqq 0,\ q''' \geqq 0.$$

*Proof.* By the Riemann-Liouville change of variables,

$$U = uq^{\frac{1}{4}} \quad \text{and} \quad ds = q^{\frac{1}{2}}(t)\,dt, \tag{2.7}$$

the differential equation (0.1) is transformed into (1.2), where $Q$ is defined by (2.1) and $t = t(s)$. The interval $T_0 < t < \infty$ is changed into some interval $(-\infty \leqq)\ S < s < S^\infty \leqq \infty$.

By Lemma 1.1 and the Remark 2, (1.2) has a pair of real-valued solutions $U = X, Y$ satisfying (1.3), as $s \to S^*$, and $|Z| \leqq 1$ for $s > S$ according as $(2.2_1)$ or $(2.2_2)$ holds. If

$$x = X/q^{\frac{1}{4}}, \qquad y = Y/q^{\frac{1}{4}} \tag{2.8}$$

(cf. (2.7)) are the corresponding solutions of (0.1), then (2.3) and the assertion concerning (2.5) follow.

**3. Monotony and convexity of $|z|$.** The main theorem of this part of the paper can be obtained as a consequence of Lemma 2.1.

THEOREM 3.1. *Let $q(t)$ satisfy the conditions and $x(t), y(t)$ the assertions of Lemma* 2.1.

(i) *Let $Q(t)$ be in case* $(2.2_1)$. *Then*

$$|z| > 0, \qquad |z|'' \geqq 0 \tag{3.1}$$

*for $t > T_0$. If, in addition, $q(t)$ is continuous for $t > 0$ and $q(t) \leqq 0$ for $0 < t \leqq T_0$, then* (3.1) *holds for $t > 0$. If $z(t)$ remains bounded as $t \to \infty$ (that is, if $q(t) \geqq$* const. $> 0$ *for large $t$), then, for $t > 0$,*

$$|z| > 0, \qquad |z|' \leqq 0, \qquad |z|'' \geqq 0. \tag{3.2}$$

(ii) *Let $Q(t)$ be in case* $(2.2_2)$. *Then, for $t > T_0$,*

$$|z| > 0, \qquad |z|' \geqq 0, \qquad |z|'' \leqq 0. \tag{3.3}$$

It is readily verified from the last part of (2.1) that if $q$ and $Q$ are monotone and $0 < q(\infty) \leqq \infty$, then $Q \to 1$ as $t \to \infty$. This gives the following

COROLLARY. *If $q(t) \leqq 0$ for $0 < t \leqq T_0$ and* (2.6) *holds for $t > T_0$, then* (3.2) *holds for $t > 0$.*

Strict inequality holds in the inequalities in (3.1), (3.2), (3.3) unless $Q \equiv 1$ for large $t$, that is, unless $q(t) \equiv (c_1 t + c_2)^{-4}$ for large $t$, where $c_1, c_2$ are constants.

If the half-line $t > T_0$ is replaced by a finite interval, then the inequalities $|z|' \leqq 0$ in (3.2) and $|z|' \geqq 0$ in (3.3) need not be valid.

*Proof of Theorem* 3.1. If $u = x(t), y(t)$ are solutions of (0.1) satisfying (2.4), then two differentiations of $r = |z| = (x^2 + y^2)^{\frac{1}{2}}$ show that

$$r'' = -qr + r^{-3} = r^{-3}(1 - qr^4). \tag{3.4}$$

In case $(2.2_1)$, the relation $(2.5_1)$ holds for $t > T_0$. It also holds for $t > T$ if $q \leqq 0$ for $T < t \leqq T_0$. Hence $r'' \geqq 0$ for all $t$ under consideration. This gives (3.1) and, hence, (3.2) if $r(t)$ is bounded as $t \to \infty$.

In the case $(2.2_2)$, the relation $(2.5_2)$ holds and implies $r'' \leqq 0$ for $t > T_0$. Since $r > 0$, this gives (3.3).

**4. Applications to Bessel functions.** The functions $v = J_\mu(t)$, $Y_\mu(t)$ are real-valued linearly independent solutions of the Bessel equation

$$v'' + v'/t + (1 - \mu^2/t^2)v = 0. \tag{4.1}$$

Hence $u = t^{\frac{1}{2}}J_\mu(t)$, $t^{\frac{1}{2}}Y_\mu(t)$ are real-valued solutions of

$$u'' + (1 - \alpha/t^2)u = 0, \text{ where } \alpha = \mu^2 - \tfrac{1}{4}. \tag{4.2}$$

Note that

$$\alpha >, =, < 0 \text{ according as } (0 \leqq)\, \mu >, =, < \tfrac{1}{2}. \tag{4.3}$$

Furthermore,

$$z = x(t) + iy(t) = (\tfrac{1}{2}\pi t)^{\frac{1}{2}} e^{i\theta}(J_\mu(t) - iY_\mu(t)), \tag{4.4}$$

where $\theta = \frac{1}{2}(\mu + \frac{1}{2})\pi$, is a solution of (4.2) satisfying

$$z = e^{it} + o(1),\ z' = ie^{it} + o(1) \text{ as } t \to \infty. \tag{4.5}$$

If (4.2) is identified with (1.2), Lemma 1.1 and the Remark 2 following it give the inequalities

$$t(J_\mu^2 + Y_\mu^2)(1 - \alpha/t^2) < 2/\pi < t(J_\mu^2 + Y_\mu^2) \text{ if } \mu > \tfrac{1}{2}, \tag{4.6}$$

$$t(J_\mu^2 + Y_\mu^2)(1 - \alpha/t^2) > 2/\pi > t(J_\mu^2 + Y_\mu^2) \text{ if } \mu < \tfrac{1}{2}, \tag{4.7}$$

for $t > 0$. Inequalities of this type were obtained by Schafheitlin [7], p. 86.

If (4.2) is identified with (0,1), so that $q = 1 - \alpha/t^2$, then the corresponding function (2.1) is

$$Q = 1 + 5\alpha^2/4(t^2 - \alpha)^3 + 3\alpha/2(t^2 - \alpha)^2 \tag{4.8}$$

for $t^2 > \max(0, \alpha)$. If $\mu > \frac{1}{2}$ (i.e., $\alpha > 0$), then $Q$ satisfies $(2.2_1)$ with $T_0 = \alpha^{\frac{1}{2}}$, so that $(2.4_1)$ is valid and becomes

$$(4.9)\qquad (t^2 - \mu^2 + \tfrac{1}{4})^{\frac{1}{2}}(J_\mu{}^2 + Y_\mu{}^2) < 2/\pi \text{ if } t > (\mu^2 - \tfrac{1}{4})^{\frac{1}{2}} \text{ and } \mu > \tfrac{1}{2}.$$

If $\mu < \frac{1}{2}$ (i.e., $\alpha < 0$), then $Q$ satisfies $(2.2_2)$ with $T = 0$, $T_1 = \frac{1}{2}(-\alpha)^{\frac{1}{2}}$ and $(2.4_2)$ gives

$$(4.10)\qquad (t^2 - \mu^2 + \tfrac{1}{4})^{\frac{1}{2}}(J_\mu{}^2 + Y_\mu{}^2) > 2/\pi \text{ if } t > 0 \text{ and } \mu < \tfrac{1}{2}.$$

It also follows from Theorem 3.1 that $r = t^{\frac{1}{2}}(J_\mu{}^2(t) + Y_\mu{}^2(t))^{\frac{1}{2}}$ satisfies $r > 0$, $r' < 0$, $r'' > 0$ or $r > 0$, $r' > 0$, $r'' < 0$ for $t > 0$ according as $\mu > \frac{1}{2}$ or $0 \leqq \mu < \frac{1}{2}$.

The inequality (4.10) contrasts sharply with the known inequality

$$(4.11)\qquad (t^2 - \mu^2)^{\frac{1}{2}}(J_\mu{}^2 + Y_\mu{}^2) < 2/\pi \text{ if } t \geqq \mu \geqq 0$$

usually deduced from Nicholson's formulae, cf. [9], p. 447. If $\mu \geqq \frac{1}{2}$, the last inequality is contained in (4.9); a derivation of (4.11) based on the results of Sections 1-3 will be indicated.

If $t = e^s$, then the Bessel equation (4.1) becomes

$$(4.12)\qquad v_{ss} + (t^2 - \mu^2)v = 0, \text{ where } t = e^s.$$

The change of variables analogous to (2.7),

$$(4.13)\qquad V = (t^2 - \mu^2)^{\frac{1}{4}}v, \qquad d\sigma = (t^2 - \mu^2)^{\frac{1}{2}}ds, \text{ where } t > \mu,$$

transforms (4.12) into an equation

$$(4.14)\qquad V_{\sigma\sigma} + RV = 0,$$

where

$$R = 1 + (e^{4s} + 4\mu^2 e^{2s})/(e^{2s} - \mu^2)^3 \text{ and } s = s(\sigma).$$

The equation has the pair of solutions $V = (t^2 - \mu^2)^{\frac{1}{4}}J_\mu$, $(t^2 - \mu^2)^{\frac{1}{4}}Y_\mu$, where $t = e^s > \mu$ and $s = s(\sigma)$. In terms of $s$, the coefficient $R$ can be written as

$$(4.15)\qquad R = 1 + (e^{-2s}/4 + \mu^2 e^{-4s})\Big(\sum_{k=0}^{\infty} \mu^{2k}e^{-2ks}\Big)^3, \quad e^s > \mu.$$

It is clear that $R$ is a decreasing function of $s$ (hence $\sigma$), so that $(1.6_2)$ in Lemma 1.1 applied to (4.14) gives (4.11).

It can also be verified from Theorem 3.1 applied to (4.14) that $r = (t^2 - \mu^2)^{\frac{1}{2}}(J_\mu{}^2 + Y_\mu{}^2)$ is increasing for $t \geqq \mu \geqq 0$; cf. [9], pp. 446-447.

The inequalities (4.9), (4.10), (4.11) suggest the examination of the

function $(t^2-\mu^2+\beta)^{\frac{1}{2}}(J_\mu{}^2+Y_\mu{}^2)$, where $\beta \geqq 0$ is a constant. Note that if the Riemann-Liouville change of variables (2.7) is altered to

$$(4.16) \qquad U=(q+\beta)^{\frac{1}{4}}u, \qquad ds=(q+\beta)^{\frac{1}{2}}dt,$$

then (0.1) is transformed into

$$(4.17) \qquad U_{ss}+[1+5q'^2/16(q+\beta)^3-q''/4(q+\beta)^2-\beta/(q+\beta)]U=0.$$

Correspondingly, if (4.13) is replaced by

$$(4.18) \qquad V=(t^2-\gamma)^{\frac{1}{4}}v, \qquad d\sigma=(t^2-\gamma)^{\frac{1}{2}}ds, \text{ where } t^2=e^{2s}>\gamma$$

and $\gamma=\mu^2-\beta$, then (4.12) is transformed into (4.14), where

$$(4.19) \qquad R=1+[e^{4s}(1-4\beta)+4\gamma e^{2s}(1+2\beta)-4\beta\gamma^2]/4(e^{2s}-\gamma)^3.$$

From the relation

$$(4.20) \qquad R_s=-[(1-4\beta)e^{6s}+2\gamma(5+4\beta)e^{4s} \\ +4\gamma^2(1-\beta)e^{2s}]/2(e^{2s}-\gamma)^4,$$

it is clear that there exists a $T=T(\beta,\mu)>0$ such that according as $\beta>\frac{1}{4}$ or $\beta<\frac{1}{4}$, $R$ is increasing or decreasing for $t \geqq T$. Thus, by Lemma 1.1, for $\mu \geqq 0$,

$$(4.21) \qquad (t^2-\mu^2+\beta)^{\frac{1}{2}}(J_\mu{}^2+Y_\mu{}^2)>2/\pi \text{ if } t>T \text{ and } \beta>\tfrac{1}{4},$$

$$(4.22) \qquad (t^2-\mu^2+\beta)^{\frac{1}{2}}(J_\mu{}^2+Y_\mu{}^2)<2/\pi \text{ if } t>T \text{ and } \beta>\tfrac{1}{4}.$$

These inequalities reduce to (4.9), (4.10) if $\beta=\frac{1}{4}$.

Obviously, $T$ can be chosen to be 0 [or $\gamma^{\frac{1}{2}}=(\mu^2-\beta)^{\frac{1}{2}}$] in (4.21) [or (4.22)] if $\beta \geqq 1$ and $\mu \leqq \beta^{\frac{1}{2}}$ (so that $\gamma \leqq 0$) [or $\beta \leqq \frac{1}{4}$ and $\mu \geqq \beta^{\frac{1}{2}}$ (so that $\gamma \geqq 0$)].

## Part II. An inhomogeneous second order equation.

**5. Preliminaries.** The familiar "alternating series argument" shows that if $f(t)$ is continuous for $t>0$, $f(t) \geqq 0$, $df(t) \leqq 0$ and $f(t) \to 0$ as $t \to \infty$, then

$$(5.1) \qquad w(t)=\int_t^\infty f(s)\sin(s-t)\,ds$$

is convergent and non-negative for $t>0$. In addition, if $f(t)$ is convex, then

$$(5.2) \qquad w'(t)=-\int_t^\infty f(s)\cos(s-t)\,ds$$

is non-positive. Furthermore, the two implications $f \geqq 0,\ f' \leqq 0 \Rightarrow w \geqq 0$ and $f \geqq 0,\ f' \leqq 0,\ f'' \geqq 0 \Rightarrow w' \geqq 0,\ w' \leqq 0$ are the first two of an infinite sequence of implications. This is clear if it is noted that (5.1) is the unique solution of

$$w'' + w = f(t) \tag{5.3}$$

satisfying $w \rightarrow 0$ as $t \rightarrow \infty$.

It will be shown below that similar facts are valid for a particular solution of the more general second order equation

$$w'' + q(t)w = f(t) \tag{5.4}$$

under suitable qualitative conditions on $q$. (The results will be extended to linear and non-linear differential equations of higher order in Part III.)

Consider the homogeneous equation belonging to (5.4),

$$v'' + q(s)v = 0. \tag{5.5}$$

It will be assumed below that (5.5) is oscillatory at $s = \infty$, i.e., that every solution $v = v(s)$ of (5.5) has infinitely many zeros custerilng at $s = \infty$.

A Green's function $G(t,s)$ for $s \geqq t$ for (5.5) will be needed. This will be defined as follows: Let $t$ be fixed. For $s \geqq t$, let $v(s) = G(t,s)$ be the solution of (5.5) determined by the initial condition,

$$G(t,s) = 0 \text{ and } G_s(t,s) = 1 \text{ if } s = t. \tag{5.6}$$

The analogues of (5.3) and (5.1) are (5.4) and

$$w(t) = \int_t^\infty G(t,s)f(s)\,ds. \tag{5.7}$$

Formally, the derivative of (5.7) is

$$w'(t) = \int_t^\infty G_t(t,s)f(s)\,ds, \tag{5.9}$$

since $G(s,s) = 0$.

The results of Part II deal with the "order of monotony" of the particular solution (5.7) of (5.4).

Many of the arguments will depend on the fact that if $v = v(s)$ is a solution of (5.5) and $q$ is positive and monotone, then (5.5) implies that

$$d(v^2 + v'^2/q) = v'^2 d(1/q), \tag{5.10}$$

$$d(qv^2 + v'^2) = v^2 dq. \tag{5.11}$$

Hence, $v^2 + v'^2$ and $qv^2 + v'^2$ are monotone. In particular, the sequence of

maxima of $|v|$ and $|v'|$ are monotone (since these maxima occur when $v'=0$ and $v=0$, respectively).

**6. Order of monotony of (5.7).** In order to state the results succinctly, it will be convenient to introduce the following terminology:

*Definition* 6.1. A function $f(t)$ will be said to be of class $M_n(a,b)$ or monotone of order $n$ on $a<t<b$ if it has $n$ ($\geqq 0$) continuous derivatives $f, f', \cdots, f^{(n)}$ satisfying

$$(6.1_j) \qquad (-1)^j f^{(j)}(t) \geqq 0$$

for $j=0,\cdots,n$ and $a<t<b$. $M_n$ will be an abbreviation for $M_n(0,\infty)$.

In all of the theorems below where it is assumed that $n>0$ and that certain functions $f, q', q_j, \cdots$ are of class $M_n(a,b)$, the assumption of the existence and continuity of the $n$-th derivative and the inequality corresponding to $(6.1_n)$ can be weakened to the assumption that the $(n-1)$-st derivative multiplied by $(-1)^{n-1}$ is non-increasing. In fact, if $n>1$, the conditions on the $(n-1)$-st and $n$-th derivatives can be replaced by the conditions that the $(n-2)$-nd derivative multiplied by $(-1)^{n-2}$ is non-decreasing and is convex.

*Definition* 6.2. $f(t)$ will be said to be of class $M_{nm}(T_0,\infty)$ if $f \in M_n(T,\infty)$ and $f$ has $m$ derivatives for $t>T_0$ satisfying

$$(6.2_j) \qquad f^{(j)}(t) \to 0 \text{ as } t\to\infty$$

for $j=0,\cdots,m$. (Note that $f\in M_{n0}(T_0,\infty)$ implies that $f\in M_{n,n-1}(T_0,\infty)$ if $n\geqq 1$. Thus the essential cases of $M_{nm}(T_0,\infty)$ are $m=0$ and $m\geqq n$.) In analogy with the above, $M_{nm}\equiv M_{nm}(0,\infty)$.

THEOREM $6.1_n$. *Let $n\geqq 0$. Let $q(t)$ have a derivative $q'(t)$ of class $M_n$ and let $0<q(\infty)\leqq\infty$. Let $f(t)$ be of class $M_{n+1,0}$. Then* (5.4) *has a unique solution* (*given by* (5.7)) *of class $M_{n,n+2}$.*

*Remark* 1. Under the assumptions on $q$, the only solution $v=v(s)$ of the homogeneous equation (5.5) satisfying $v'(s)\to 0$ as $s\to\infty$ is $v\equiv 0$. (In fact, by (5.11), the successive maxima of $|v'(s)|$ are non-decreasing.) Hence, (5.4) has at most one solution satisfying $w'\to 0$ as $t\to\infty$. Also, if $0<q(\infty)<\infty$, then (5.4) has at most one solution satisfying $w\to 0$ as $t\to\infty$.

*Remark* 2. If all solutions of (5.5) tend to 0 as $t\to\infty$ (e.g., if "$q(t)\to\infty$

smoothly as $t \to \infty$," cf. [2]), then the condition $f(\infty) = 0$ implicit in $f \in M_{n+1,0}$ can be omitted in Theorem $6.1_n$ except for assertion $w'' \to 0$ when $n = 0$.

*Remark* 3. Suppose that there is a $t = T_0$, $0 \leqq T_0 < \infty$, such that $q \leqq 0$ or $q > 0$ according as $0 < t \leqq T_0$ or $t > T_0$. For $n > 0$ and $0 < t < T_0$, the conditions on $q$ and $f$ can then be lightened somewhat: for $n = 1$ and $n = 2$, it is sufficient to require only $q \leqq 0$, $f \geqq 0$; for $n > 2$, it is sufficient to require that $q', f \in M_{n-2}(0, T_0)$. If the assertion $w \in M_{n,n+2}$ is weakened to $w \in M_{n,2}$ for $n = 1, 2$, the conditions can also be lightened for $t > T_0$: for $n = 1$, it is sufficient to require that $q$ is non-decreasing, $f \geqq 0$ is non-increasing with $f(\infty) = 0$ and $f/q$ is convex; for $n = 2$, it is sufficient to impose the additional condition that $-(f/q)'$ is convex.

*Remark* 4. The proof of Theorem $(6.1_0)$ will give the following a priori bounds for $w$ and $w'$:

$$(6.3) \qquad 0 \leqq w(t) \leqq \pi f(t)/q(t), \qquad |w'(t)| \leqq \pi f(t)/q^{\frac{1}{2}}(t);$$

cf. (7.6) below. The upper bound for $w$ is improved successively to $2f(t)/q(t)$ and $f(t)/q(t)$ in the proofs of Theorems $6.1_1$ and $6.1_2$; cf. (8.4) and (9.2).

Theorem $6.1_n$ and the theorem of Hausdorff-Bernstein imply the following:

COROLLARY. *Let $q, f$ satisfy the conditions of Theorem $6.1_n$ for $n = 1, 2, \cdots$, then the solution* (5.7) *of* (5.4) *has a representation as a Laplace-Stieltjes integral*

$$(6.4) \qquad w(t) = \int_0^\infty e^{-st} d\sigma(s) \text{ for } t > 0$$

*with some non-decreasing weight function $\sigma = \sigma(s)$.*

Theorems $6.1_0$, $6.1_1$, $6.1_2$ and the corresponding Remark 3 will be proved in Sections 7, 8, 9 below. Theorem $6.1_n$ with $n > 2$ is more subtle and does not seem to follow from the cases $n \leqq 2$ by successive differentiation. In fact, its proof will involve a new existence proof for each $n$ for a differential equation of order $n - 2$. Theorem $6.1_n$, with $n > 2$, will be proved in Sections 16-17 in Part III below.

Theorem $6.1_n$ deals with the case of a non-decreasing $q$. In order to state analogous theorems for the case of a non-decreasing $q$, it will be convenient to introduce the following definition:

*Definition* 6.3. Let the classes of functions $DM_n(a, b)$, $DM_n$, $DM_{nm}(T_0, \infty)$, $DM_{nm}$ be defined as the analogues of $M_n(a, b)$, $M_n$, $M_{nm}(T_0, \infty)$,

$M_{nm}$ in which the $j$-th derivative $f^{(j)} = d^jf/dt^j$ on $(6.1_j)$ and/or $(6.2_j)$ is replaced by $D^{(j)}f$, where $D$ is the differential operator $q^{-1}(t)d/dt$.

THEOREM $6.2_n$. *Let $n \geqq 0$. Let $q(t)$ be continuous, non-increasing and let (5.5) be oscillatory at $\infty$ (in particular, $q > 0$). Let $1/q^2$ have a derivative of class $DM_n$. Let $f/q$ be of class $DM_{n+1,0}$. Then (5.4) has a unique solution $w = w(t)$ (given by (5.7)) of class $M_{0,2}$ and, if $n > 0$, then $-w'$ is of class $DM_{n-1,n+1}$.*

Theorem $6.2_0$ will be proved in Section 10. Theorem $6.2_n$, with $n > 0$, will be deduced from Theorem $6.1_{n-1}$ in Section 11.

**7. Proof of Theorem $6.1_0$.** Note that $q(s) \geqq q(t) > 0$ for $s \geqq t > T_0$. Thus, if $v = v(s) \not\equiv 0$ is a solution of (5.5), then the graph of $y = |v(s)|$ consists of a sequence of "arches." The first and second comparison theorems of Sturm imply that if the $k$-th arch is over the interval $s_k \leqq s \leqq s_{k+1}$ and $q(s_0) \leqq q(s^0)$ when $s_{k-1} \leqq s_0 \leqq s_k \leqq s^0 \leqq s_{k+1}$, then a reflection of the $k$-th arch across the line $s = s_k$ gives an arch lying under the $(k-1)$-st arch (i.e., $|v(s_k + t)| \leqq |v(s_k - t)|$ for $0 \leqq t \leqq s_{k+1} - s_k$ and $s_{k+1} - s_k \leqq s_k - s_{k-1}$); cf. [4], p. 531 and p. 538. Thus if $f \in M_{10}$, the "alternating series argument" implies that (5.7) is convergent and non-negative for $t > 0$.

The kernel $G(t,s)$, for $s \geqq t$, can be written in the form

$$G(t,s) = v_1(t)v_2(s) - v_1(s)v_2(t), \tag{7.1}$$

where $v = v_1(s), v_2(s)$ are arbitrary solutions of (5.5) subject to the Wronskian condition

$$v_1(s)v_2'(s) - v_1'(s)v_2(s) \equiv 1. \tag{7.2}$$

Also, for fixed $t$,

$$G_t(t,s) = v_1'(t)v_2(s) - v_1(s)v_2'(t) \tag{7.3}$$

is a linear combination of $v_1(s), v_2(s)$ and is, therefore, a solution of (5.5). Hence, the integral in (5.9) is convergent (for the same reasons as is the integral in (5.7)).

If the upper limit of integration $\infty$ is replaced by a fixed $T$, it is seen that (5.7) and (5.9) represent a solution of (5.4) and its derivative. Thus the same is true for (5.7), (5.9) if it is verified that the integrals are uniformly convergent on compact $t$-sets of $t > 0$.

By the alternating series argument, the remainder $\left|\int_T^\infty\right|$ of the integral

in (5.7) or (5.9) is majorized by $C(b-a)f(T)$, where $s=a, b$ are successive zeros of $v(s)=G(t,s)$ or $v(s)=G_t(t,s)$, $a\leqq T\leqq b$, and $|G|$ or $|G_t|$ is majorized by $C$ for $s\geqq t$. The monotony of $q$ implies that $b-a\leqq \pi/q^{\frac{1}{2}}(t)$. By (5.10), $v(s)=G(t,s)$ is such that $V=v^2+v'^2/q$ is non-increasing for $s>t$. Hence $v^2(s)\leqq V(s)\leqq V(t)$ for $s\geqq t>T_0$. Thus

$$(7.4) \qquad |G(t,s)|^2\leqq |G_s(t,t)|^2/q(t) = 1/q(t) \text{ for } s\geqq t>T_0.$$

Similarly (cf. (5.8)),

$$(7.5) \qquad |G_t(t,s)|^2\leqq |G_t(t,t)|^2=1 \text{ for } s\geqq t>T_0.$$

Hence $C$ can be chosen to be $1/q^{\frac{1}{2}}(t)$, 1 in the respective cases of (5.7), (5.9). Thus the remainder $\left|\int_T^\infty\right|$ is at most $\pi f(T)/q(t)$ or $\pi f(T)/q^{\frac{1}{2}}(t)$ in the cases (5.7) or (5.9). This proves the uniform convergence of (5.7), (5.9) on compact subsets of $t>0$.

This argument (with $T=t$) also shows that if $t>T_0$, then

$$(7.6) \qquad 0\leqq w(t)\leqq \pi f(t)/q(t), \qquad |w'(t)|\leqq \pi f(t)/q^{\frac{1}{2}}(t).$$

Hence $w, w'\to 0$ as $t\to\infty$. Also $w''=-qw+f$ satisfies $|w''|\leqq(\pi+1)f(t)\to 0$ as $t\to\infty$. This proves Theorem 6.$1_0$.

**8. Proof of Theorem 6.$1_1$.** In order to prove $w'(t)\leqq 0$ for $t>0$, consider first only $t>T_0$ and write (5.9) as

$$(8.1) \qquad w'(t)=\int_t^\infty [G_t(t,s)q(s)][f(s)/q(s)]ds.$$

Since $v(s)=G_t(t,s)$ is a solution of (5.5), it follows that $G_t(t,s)q(s) = -G_{tss}(t,s)$. An integration by parts applied to (8.1) gives

$$w'(t)=-G_{ts}(t,s)f(s)/q(s)]_{s=t}^{s=\infty}+\int_t^\infty G_{ts}(t,s)\,(f/q)'ds.$$

The integrated terms vanish for, on the one hand, (7.3) show that $G_{ts}(t,t)=0$, and, on the other hand, $G_{ts}(t,s)/q^{\frac{1}{2}}(s), 1/q^{\frac{1}{2}}(s)=O(1)$ and $f(s)\to 0$ as $s\to\infty$. (The boundedness of $G_{ts}(t,s)/q^{\frac{1}{2}}(s)$ follows from the non-increasing character of $v^2+v'^2/q$, where $v(s)=G_t(t,s)$; cf. (5.10).)

Another integration by parts gives

$$w'(t)=(G_s(t,s)+1)\,(f(s)/q(s))']_{s=t}^{s=\infty}-\int_t^\infty [G_t(t,s)+1]\,(f/q)''ds.$$

The integrated term at $s=t$ vanishes for $G_t(t,t)+1=0$ by (7.2), (7.3).

The term at $s=\infty$ is also 0 for $v(s)=G_t(t,s)$ is bounded and $(f/q)'\to 0$ as $s\to\infty$. (The last limit relation follows from the fact that $(f/q)'$ is monotone and integrable over $t\leqq s<\infty$.) Thus

$$(8.2)\qquad w'(t)=-\int_t^\infty [G_t(t,s)+1](f(s)/q(s))''ds.$$

The assumptions on $f, q$ imply that $(f/q)''\geqq 0$ for $s\geqq T_0$. The factor $G_t(t,s)+1\geqq 0$ by (7.5). Thus $w'(t)\leqq 0$ for $t>T_0$.

In order to deal with $w'(t)$ on the interval $0<t\leqq T_0$, when $T_0>0$, note that if a solution of (5.4) (i.e., $w''=-qw+f$) satisfies initial conditions $w(T_0)>0$, $w'(T_0)<0$, then $q\leqq 0$ and $f\geqq 0$ imply, for reasons of convexity, that $w(t)\geqq 0$, $w'(t)\leqq 0$, $w''(t)\geqq 0$ for $0<t\leqq T_0$. By continuity, the same holds for initial conditions $w(T_0)\geqq 0$, $w'(T_0)\leqq 0$. Hence $w'(t)\leqq 0$ for $0<t\leqq T_0$.

It remains to show that $w'''(t)\to 0$ as $t\to\infty$. Note that the first factor in the integrand in (8.2) satisfies $|G_t(t,s)+1|\leqq 2$ for $t>T_0$, by (7.5). Hence (8.2) implies that

$$(8.3)\qquad 0\leqq -w'(t)\leqq -2(f(t)/q(t))' \text{ for } t>T_0.$$

Thus, the first inequality in (7.6) can be improved to

$$(8.4)\qquad 0\leqq w(t)\leqq 2f(t)/q(t) \text{ for } t>T_0.$$

By (8.3) and the boundedness of $q'$,

$$(8.5)\qquad 0\leqq -qw'\leqq -2f'+2fq'/q\to 0 \text{ as } t\to\infty.$$

Thus a differentiation of (5.4) shows that

$$(8.6)\qquad w'''=-qw'-q'w+f'\to 0 \text{ as } t\to\infty.$$

This proves Theorem $6.1_1$.

**9. Proof of Theorem $6.1_2$.** It will first be shown that $w''(t)\geqq 0$. If $q\leqq 0$ for $0<t\leqq T_0$, then $w''=-qw+f\geqq 0$ for $0<t\leqq T_0$. Thus, it can be supposed that $t>T_0$ and that $q(t)>0$ on this $t$-range. Dividing (5.4) by $q$ and differentiating twice gives

$$(9.1)\qquad W''+qW=(f/q)'', \text{ where } W=w''/q.$$

Since $q$ satisfies the conditions of Theorem $6.1_2$, hence Theorem $6.1_0$, and $(f/q)''\in M_{1,0}(T_0,\infty)$, it follows from Theorem $6.1_0$ that (9.1) has a non-negative solution for $t>T_0$ given by

$$W(t)=\int_t^\infty G(t,s)(f/q)''ds$$

satisfying $W, W', W'' \to 0$ as $t \to \infty$. By uniqueness (cf. Remark 1 following the statement of Theorem $6.1_n$), this solution is $W = w''/q$, where $w$ is given by (5.7).

(Note that under the conditions of Theorem $6.1_2$, the inequality in (8.4) can be improved to

$$(9.2) \qquad 0 \leqq w(t) \leqq f(t)/q(t) \text{ for } t > T_0.$$

This is clear from the inequality $w'' = f - qw \geqq 0$.)

It remains to show that $w^{(4)}(t) \to 0$ as $t \to \infty$. Let (5.4) be differentiated to give

$$(9.3) \qquad w''' + qw' = f' - q'w.$$

The function $-(f' - q'w)$ is of class $M_{2,0}(T_0, \infty)$. Thus, by Theorem $6.1_1$, equation (9.3) considered as a second order, inhomogeneous equation for $w'$ has a unique solution, the negative of which is class $M_{1,3}(T_0, \infty)$. Since uniqueness refers to the class of solutions satisfying $(w')' \to 0$ as $t \to \infty$, this solution is the derivative of (5.7). Thus $-w' \in M_{1,3}$ and $w \geqq 0$, that is, $w \in M_{2,4}$. This proves Theorem $6.1_2$.

**10. Proof of Theorem $6.2_0$.** Since $v(s) = G(t,s)$ is a solution of (5.5), $G(t,s) = -G_{ss}(t,s)/q(s)$. Thus, without considering questions of convergence, the integral in (5.7) can be written formally as

$$(10.1) \qquad w(t) = -\int_t^\infty G_{ss}(t,s) f(s)\, ds/q(s).$$

An integration by parts gives

$$(10.2) \qquad w(t) = (1 - G_s(t,s))(f(s)/q(s))]_{s=t}^{s=\infty} - \int_t^\infty [1 - G_s(t,s)](f/q)'ds.$$

The integrated term at $s = t$ vanishes since $1 - G_s(t,t) = 0$ by (6.2). Since $q$ is non-increasing, (5.11) shows that $V = qv^2 + v'^2$ is non-increasing if $v(s) = G(t,s)$. In particular, $v'^2(s) \leqq V(s) \leqq V(t) = 1$ for $s \geqq t$ and so, $|G_s(t,s)| \leqq 1$. Thus

$$(10.3) \qquad 0 \leqq 1 - G_s(t,s) \leqq 2 \text{ for } s \geqq t.$$

Since $f/q \to 0$ and $s \to \infty$, the integrated term at $s = \infty$ in (10.2) is 0. Since $(f/q)' \leqq 0$ and is (absolutely) integrable over $t \leqq s < \infty$, it follows that the integral in (10.1) is convergent and

$$(10.4) \qquad w(t) = -\int_t^\infty [1 - G_s(t,s)](f(s)/q(s))'ds.$$

In fact, the integral in (10.1) is uniformly convergent on compact subsets of $t > 0$.

Since $1 - G_s(t,s) = 0$ if $s = t$, differentiation of (10.4) gives

$$w'(t) = \int_t^\infty G_{ts}(t,s)(f(s)/q(s))'ds. \tag{10.5}$$

This interchange of differentiation and integration is valid for the monotony of $q$ and (5.11) imply that

$$|G_{ts}(t,s)| \leqq q^{\frac{1}{2}}(t) \leqq \text{const. for } s \geqq t. \tag{10.6}$$

It is now readily verified that (5.7) represents a solution of (5.4). Also (10.3) and (10.4) show that $w(t) \geqq 0$ and

$$0 \leqq w(t) \leqq 2f(t)/q(t) \text{ for } t > 0. \tag{10.7}$$

Also, (10.5) and (10.6) give

$$|w'(t)| \leqq f(t)/q^{\frac{1}{2}}(t) \text{ for } t > 0. \tag{10.8}$$

These facts imply the assertions of Theorem $6.2_0$, namely, $w(t) \geqq 0$ and $w, w', w'' \to 0$ as $t \to \infty$.

**11. Proof of Theorem $6.2_n$, $n > 0$.** Dividing (5.4) by $q$ and differentiating gives

$$(w''/q)' + w' = (f/q)'. \tag{11.1}$$

On introducing the new variables $W$, $\tau$ defined by

$$W = -w' \text{ and } d\tau = q(t)dt \tag{11.2}$$

(11.1) can be written as

$$D^2W + (1/q)W = -D(f/q), \text{ where } D = d/d\tau = q^{-1}(t)d/dt. \tag{11.3}$$

Note that $(1/q^2)' = 2D(1/q)$. Thus an assumption of Theorem $6.2_n$ implies that $D(1/q) \in DM_n$. Since $n > 0$, $D(1/q) \geqq 0$ and $D^2(1/q) \leqq 0$, as a function of $\tau$, $1/q$ is non-decreasing and concave. Since (11.3) is oscillatory at the upper end of the $\tau$-range, it follows that $\tau(t)$ satisfies $\tau(\infty) = \infty$. Thus, $0 < t < \infty$ is mapped onto some range $(-\infty \leqq) T^0 < \tau < \infty$.

The assumptions of Theorem $6.2_n$ imply that, as functions of $\tau$, $1/q$ and $-D(f/q)$ satisfy the assumptions of Theorem $6.1_{n-1}$. Hence, (11.3) has a unique solution $W = W(t)$ of class $DM_{n-1,n+1}$.

Uniqueness refers to the class of solutions satisfying $DW \to 0$ as $t \to \infty$. According to the last section (11.2), the function $w$ given by (5.7) is such

that $W = -w'$ is a solution of (11.3). Also, $DW = -w''/q = w - f/q \to 0$ as $t \to \infty$. Thus, if $w(t)$ is defined by (5.7), then $W = -w' \in DM_{n-1,n+1}$. This proves Theorem 6.$2_n$.

## Part III. Differential equations of higher order.

**12. Statement of results.** The object of this part of the paper is to obtain analogues of Theorem 6.$1_n$ for differential equations of the form

$$w^{(k+2)} + q(t)w^{(k)} - \sum_{j=0}^{k-1}(-1)^{j+k}g_j(t)w^{(j)} = (-1)^k f(t) \tag{12.1}$$

and for related non-linear equations of order $k+2$.

THEOREM 12.$1_n$. *Let $n \geqq 0$. Let $q(t)$ posssess a derivative $q'(t)$ of class $M_n$ and $0 < q(\infty) \leqq \infty$. Let $f(t) \in M_{n+1,0}$. If $k \geqq 1$, assume*

$$\int^{\infty} t^{k-1}f(t)\,dt/q(t) < \infty, \tag{12.2}$$

*$g_j(t) \in M_{n+1}$, and*

$$\int^{\infty} t^{k-j-1}g_j(t)\,dt/q(t) < \infty \quad \text{for } j = 0, \cdots, k-1. \tag{12.3}$$

*Then* (12.1) *has a unique solution $w = w(t)$ of class $M_{n+k,n+k+2}$.*

*Remark* 1. If, in addition, $g_0(\infty) = 0$ and $c$ is a positive constant, then (12.1) has a unique solution $w = w(t)$ such that $w - c \in M_{n+k,n+k+2}$. This follows by introducing $w - c$ as a new dependent variabile in (12.1).

*Remark* 2. If all solutions of (5.5) tend to 0 as $t \to \infty$, then the conditions $f(\infty) = 0$ in Theorem 12.$1_n$ and the condition $g_0(\infty) = 0$ in the last remark can be omitted except for assertion $w^{(k)} \to 0$ when $n = 0$.

*Remark* 3. If there is a $T_0 > 0$ such that $q(t) \leqq 0$ for $0 < t < T_0$ and $q(t) > 0$ for $t > T_0$, the conditions on $q$, $f$, $g_j$ can be reduced on the interval $0 < t < T$ corresponding to Remark 3 following Theorem 6.$1_n$: for $n = 1, 2$, it is sufficient to require only $q \leqq 0$, $f \geqq 0$, $g_j \geqq 0$; for $n > 2$, it is sufficient to require that $q', f, g_j \in M_{n-2}(0, T_0)$.

Theorem 12.$1_n$ and its Corollary should be contrasted with analogous theorems in [5], Appendix, in which it is assumed that $f \equiv 0$ and that $v'' + q(s)v = 0$ is disconjugate (rather than oscillatory) for $s > 0$, but no conditions like (12.2)-(12.3) occur.

*Remark* 4. It should be noted that neither of the conditions (12.2), (12.3) can be omitted in Theorem $12.1_n$ for any $n$ (if $k \geqq 1$).

In order to see this, consider first the differential equation

$$w''' + w' = -1/(t+1),$$

where $k = 1$, $q = 1$, $g = 0$ and $f = (t+1)^{-1}$. All conditions of Theorem $12.1_n$ hold for every $n$ except (12.2). If $w = w(t)$ is a solution for which $w' \leqq 0$, then

$$w'(t) = -\int_t^\infty (s+1)^{-1} \sin(s-t)\, ds,$$

by the Remark 1 following Theorem $6.1_n$. It is easily seen, from an integration by parts, that $w'(t) = -t^{-1} + O(t^{-2})$ as $t \to \infty$. Hence $w(t) \to -\infty$ as $t \to \infty$. Consequently, (12.2) cannot be omitted in Theorem $12.1_n$.

The differential equation

$$w''' + w' + Cw/t \log t = -2\epsilon/t \log^2 t, \tag{12.4}$$

where $C > 1$ and $\epsilon > 0$, satisfies all conditions of Theorem $12.1_n$ for every $n$ except (12.3) if $t > 0$ is replaced by $t > 2$. If this equation has a solution $w = w(t)$ of class $M_1(2, \infty)$, then its derivative satisfies

$$\begin{aligned} -w' = 2\epsilon \int_t^\infty & \sin(s-t)\, ds/s \log^2 s \\ & + C \int_t^\infty w(s) \sin(s-t)\, ds/s \log s. \end{aligned} \tag{12.5}$$

Let $T$ be so large that $t \geqq T$ implies the inequality

$$C \int_t^\infty \sin(s-t)\, ds/s \log^2 s \geqq 1/t \log^2 t$$

and the inequality which results if $C$ is replaced by $2\epsilon$ and the 1 on the right by $\epsilon$. Then, since the last term of (12.5) is non-negative, $-w' \geqq \epsilon/t \log^2 t$ for $t \geqq T$. Hence $w = \epsilon/t \log t +$ (a non-negative, non-increasing function). Thus (12.5) implies $-w' \geqq 2\epsilon/t \log^2 t$ and so, $w(t) = 2\epsilon/t \log t +$ (a non-negative, non-increasing function). Continuing this argument, one obtains the contradiction $w(t) \geqq m\epsilon/t \log t$ for $t \geqq T$ and $m = 1, 2, \cdots$.

Actually, conditions (12.2), (12.3) can be lightened somewhat.

THEOREM $12.2_n$. *Let $q, g_0, \cdots, g_{k-1}, f$ satisfy the conditions of Theorem $12.1_n$ except that (12.2)-(12.3) is replaced by the following when $k \geqq 1$: There exist positive, continuous functions $\epsilon_1(t), \cdots, \epsilon_k(t)$ for large $t$ with the properties that, as $t \to \infty$,*

$$\int_t^\infty s^{m-1}q^{-1}(s)f(s)\,ds = O(\epsilon_m(t)), \tag{12.6}$$

$$\sum_{j=0}^{k-1}\int_t^\infty s^{m-1}q^{-1}(s)g_j(s)\epsilon_{k-j}(s)\,ds = o(\epsilon_m(t)), \tag{12.7}$$

*for* $m=1,\cdots,k$. *Then* (12.1) *has a unique solution* $w=w(t)$ *of class* $M_{n+k,n+k+2}$ *satisfying* $w^{(j)}(t)=O(\epsilon_{k-j}(t))$ *as* $t\to\infty$ *for* $j=0,\cdots,k-1$.

When (12.2)-(12.3) hold, then (12.6)-(12.7) are satisfied by $\epsilon_j(t)=1/t^{k-j}$ for $j=1,\cdots,k$. Also, if $w=w(t)$ is any function of class $M_k$, then $w^{(j)}(t)=O(1/t^{k-j})$ as $t\to\infty$ for $j=0,\cdots,k-1$. Thus Theorem 12.$2_n$ is contained in Theorem 12.$3_n$.

*Remark* 5. On the one hand, Theorem 12.$2_n$ becomes false if "$o(\epsilon_m(t))$" in (12.7) is replaced by "$O(\epsilon_m(t))$." This can be seen from the example (12.4) where $k=1$, $q=1$, $g_0=C/t\log t$ and $f=2\epsilon/t\log^2 t$, so that the integral in (12.7) is majorized by $C\epsilon_1(t)$ if $\epsilon_1(t)=1/\log t$. On the other hand, (12.6)-(12.7) can be weakened to

$$\gamma_n\int_t^\infty (s-t)^{m-1}q^{-1}(s)f(s)\,ds \leqq (1-\theta)\epsilon_m(t), \tag{12.8}$$

$$\int_t^\infty (s-t)^{m-1}G(s)\,ds \leqq \theta\epsilon_m(t), \tag{12.9}$$

for $t\geqq T$ and $m=1,\cdots,k$, where $0<\theta<1$, $T$ is some (fixed) positive number,

$$G(s)=\gamma_n q^{-1}(s)\sum_{j=0}^{k-1} g_j(s)\epsilon_{k-j}(s)/(k-j-1)!, \tag{12.10}$$

$\gamma_0=\pi$, $\gamma_1=2$ and $\gamma_n=1$ for $n\geqq 2$. It is clear that (12.6)-(12.7) imply (12.8)-(12.9) if $\epsilon_m(t)$ is replaced by (const.)$\epsilon_m(t)$ for a suitable constant. The factor $\gamma_n=1$ in (12.9)-(12.10) for $n\geqq 2$ cannot be replaced by a smaller constant (even when the constant implicit in the $O$-term of (12.6) is arbitrarily small). For example, (12.4) has a completely monotone solution if $\epsilon>0$ and $0\leqq C<1$, but it has no non-negative, non-increasing solution if $\epsilon>0$, $C>1$.

*Remark* 6. Under the assumptions of Theorem 12.$2_n$, (12.1) can have more than one solution of class $M_{n+k,n+k+2}$. For Theorem 20.$1_n$ below implies that the differential equation (18.2) has a non-trivial solution $w=w(t)\not\equiv 0$, as well as the trivial solution $w\equiv 0$, of class $M_{n+1,n+3}$. In (18.2), where $k=1$, $f=0$ and $q(\infty)=\infty$, the function $\epsilon_1(t)=e^{-t}$ satisfies (12.7) since $q'/q\to 0$, $t\to\infty$.

Theorem 12.$2_n$ has an analogue for non-linear equations of the form

$$(12.11) \qquad w^{(k+2)} + q(t)w^{(k)} = (-1)^k F(t, w, -w', \cdots, (-1)^{k-1}w^{(k-1)}).$$

Introduce the abbreviations

$$(12.12) \qquad f(t) = F(t, 0, \cdots, 0)$$

and, if $F = F(t, \alpha_0, \cdots, \alpha_{k-1})$,

$$(12.13) \qquad g_j(t) = \partial F/\partial\alpha_j \text{ at } \alpha_i = \epsilon_{k-i}(t)/(k-i-1)! \text{ for } j = 0, 1, \cdots, k-1.$$

THEOREM 12.$3_n$. *Let $q(t)$ be as in Theorem 12.$1_n$ and $q(t) > 0$ for $t > 0$. Let $F(t, \alpha_0, \cdots, \alpha_{k-1})$ be defined for $t > 0$, $\alpha_i \geqq 0$ and have continuous partial derivatives satisfying*

$$(12.14) \qquad (-1)^m \partial^{m+h+i+\cdots+j} F/\partial t^m \partial\alpha_0{}^h \partial\alpha_1{}^i \cdots \partial\alpha_{k-1}{}^j \geqq 0$$

*for $m \leqq n+1$ and $m + h + i + \cdots + j \leqq n + 2$; let*

$$(12.15) \qquad F(t, \alpha_0, \cdots, \alpha_{k-1}) \to 0 \text{ as } (t, \alpha_0, \cdots, \alpha_{k-1}) \to (\infty, 0, \cdots, 0);$$

*finally, let there exist positive continuous functions $\epsilon_1(t), \cdots, \epsilon_k(t)$ for $t > 0$ and a constant $\theta, 0 < \theta < 1$, such that* (12.12), (12.13), (12.10) *satisfy* (12.8), (12.9) *for $t > 0$ and $m = 1, \cdots, k$. Then* (12.10) *has a unique solution $w = w(t)$ of class $M_{n+k,n+k+2}$ satisfying $(-1)^j w^{(j)}(t) \leqq \epsilon_{k-j}(t)/(k-j)!$ for $t > 0$ and $j = 0, \cdots, k-1$.*

Combining Theorems 12.$1_n$, 12.$2_n$ or 12.$3_n$ with the theorem of Hausdorff-Bernstein gives

COROLLARY. *If the conditions of Theorem 12.$1_n$ [or 12.$2_n$ or 12.$3_n$] are satisfied for $n = 0, 1, \cdots$, then* (12.1) *[or* (12.1) *or* (12.11)*] has a unique [or at least one] solution $w = w(t)$ representable in the form*

$$w(t) = \int_0^\infty e^{-st} d\sigma(s) \text{ for } t > 0, \text{ where } d\sigma \geqq 0.$$

*If, in addition, to the conditions of Theorem 12.$1_n$, $g_0(\infty) = 0$ and $c > 0$, then* (12.1) *has a unique solution representable in the form*

$$w(t) = c + \int_0^\infty e^{-st} d\sigma(s) \text{ for } t > 0, \text{ where } d\sigma \geqq 0.$$

The existence statement of Theorems 12.$2_n$ and 12.$3_n$, for the cases $n = 0, 1, 2$, will be proved in Section 13, the uniqueness statement in Section 14. This will be used to prove Theorem 6.$1_n$ for $n > 2$, first for the case

$q(\infty) < \infty$ in Section 15 and then for the case $q(\infty) = \infty$ in Section 16. These results, in turn, will be used to complete the proof of Theorems 12. $2_n$ and 12. $3_n$. $n > 2$, in Section 17.

**13. Existence in Theorems 12. $2_n$ and 12. $3_n$, $n \leqq 2$.** In this section, it will be supposed that $n = 0$, 1, or 2. If $k = 0$, the assertion to be proved is contained in Theorem 6. $1_n$. Suppose therefore that $k \geqq 1$.

In order to deal with Theorems 12. $2_n$ and 12. $3_n$ at the same time, let (12.1) be written in the form (12.11). For the proof of Thetorem 12. $3_n$, it will be supposed that $T$ is any positive number; for Theorem 12. $2_n$, it will be supposed that $T$ is so large that $q(t) > 0$ and (12.8), (12.9) hold for $t \geqq T$.

The desired solution will be obtained by successive approximations. Let $w_0 \equiv 0$. If $w_m(t)$ have been defined, put

$$(13.1_m) \qquad f_m(t) = F(t, w_m(t), -w_m'(t), \cdots, (-1)^{k-1} w_m^{(k-1)}(t)).$$

If possible, define the $k$-th derivative of $w_{m+1}$ by

$$(13.2_m) \qquad w_{m+1}^{(k)}(t) = (-1)^k \int_t^\infty G(t, s) f_m(s)\, ds$$

and the lower order derivatives by

$$(13.3_m) \qquad w_{m+1}^{(j)}(t) = (-1)^{k-j} \int_t^\infty (s-t)^{k-j-1} w_{m+1}^{(k)}(s)\, ds/(k-j-1)!$$

for $j = 0, \cdots, k-1$. Thus, formally

$$(13.4_m) \qquad w_{m+1}^{(k+2)} + q w_{m+1}^{(k)} = (-1)^k f_m.$$

It will be shown, by induction on $m$, that $w_0 \equiv 0$ and (13.1)-(13.4) define a sequence of functions $w_0, w_1, \cdots$ satisfying

$$(13.5_m) \qquad w_m \in M_{n+k,n+k+2}, \qquad f_m \in M_{n+1,0},$$

in fact, $\Delta w_m = w_m - w_{m-1}$ satisfies

$$(13.6_m) \qquad \Delta w_{m+1} \in M_{n+k,n+k+2},$$

$$(13.7_m) \qquad (-1)^j \Delta w_{m+1}^{(j)} \leqq (1-\theta)\theta^m \epsilon_{k-j}(t)/(k-j-1)! \text{ for } t \geqq T, \quad j = 0, 1, \cdots, k-1.$$

Note that, by $(13.1_m)$ and (12.14),

$$(13.8) \qquad w_m \in M_{n+k,n+k+2} \text{ implies } f_m \in M_{n+1,0}.$$

If $w_m, w_{m+1}$ have been defined and $\Delta f_m = f_m - f_{m-1}$, then

$$(13.9) \qquad \Delta f_{m+1}(t) = \sum_{j=0}^{k-1} (-1)^j \left( \int_0^1 F_j{}^0 d\tau \right) \Delta w_{m+1}{}^{(j)}(t),$$

where $F_j = \partial F / \partial \alpha_j$ and

$$(13.10) \qquad \begin{aligned} F_j{}^0(t,\tau) = F_j(t, [\tau w_{m+1}(t) + (1-\tau) w_m], \\ \cdots, (-1)^{k-1}[\tau w_{m+1}{}^{(k-1)}(t) + (1-\tau) w_m{}^{(k-1)}(t)]). \end{aligned}$$

Thus, by (12.14),

(13.11) $\quad$ $(13.5_m)$, $(13.5_{m+1})$, $(13.6_m)$ imply $\Delta f_{m+1} \in M_{n+1,0}$.

If $(13.7_0), \cdots, (13.7_m)$ hold, then

$$(-1)^j w_{m+1}{}^{(j)} \leqq (1-\theta) \sum_{i=0}^{m} \theta^m \epsilon_{k-j}/(k-j-1)! \leqq \epsilon_{k-j}/(k-j-1)!.$$

Also, if $(-1)^j \Delta w_{m+1}{}^{(j)} \geqq 0$, then

$$(-1)^j [\tau w_{m+1}{}^{(j)} + (1-\tau) w_m{}^{(j)}] \leqq (-1)^j w_{m+1}{}^{(j)} \text{ for } 0 \leqq \tau \leqq 1.$$

Thus, by (13.9), (12.13) and (12.10),

(13.12) $\quad$ $(13.6_m)$ and $(13.7_0), \cdots, (13.7_m)$ imply $(13.13_m)$,

where

$$(13.13_m) \qquad \gamma_n \Delta f_{m+1}/q \leqq (1-\theta)\theta^m G \text{ for } t \geqq T.$$

If $w_{m-1}$, $w_m$ have been defined for some $m \geqq 1$, then the $k$-th derivative of $w_{m+1}$ or, equivalently, of $\Delta w_{m+1}$ will be defined by the use of

$$(13.14) \qquad \Delta w_{m+1}{}^{(k+2)} + q \Delta w_{m+1}{}^{(k)} = (-1)^k \Delta f_m.$$

If $(13.5_{m-1})$, $(13.5_m)$ and $(13.6_{m-1})$ hold, so that $\Delta f_m \in M_{n+1,0}$ by (13.11), then (13.14) considered as a second order equation for $\Delta w_{m+1}{}^{(k)}$ has, according to Theorem $6.1_n$, a unique solution such that $(-1)^k \Delta w_{m+1}{}^{(k)} \in M_{n,n+2}$ and

$$(13.15) \qquad \Delta w_{m+1}{}^{(k)} = (-1)^k \int_t^\infty G(t,s) \Delta f_m(s)\, ds.$$

Also, the remark concerning (6.3) shows that for $t \geqq T$,

$$(13.16) \qquad |\Delta w_{m+1}{}^{(k)}| \leqq \gamma_n \Delta f_m / q, \qquad |\Delta w_{m+1}{}^{(k)\prime}| \leqq \pi \Delta f_m / q^{\frac{1}{2}}.$$

If, in addition, $(13.13_{m-1})$ holds, then the integrals in

$$(13.17) \qquad \begin{aligned} &\Delta w_{m+1}{}^{(j)}(t) \\ &= (-1)^{k-j} \int_t^\infty (s-t)^{k-j-1} \Delta w_{m+1}{}^{(k)}(s)\, ds/(k-j-1)! \end{aligned}$$

for $j = 0, \cdots, k-1$ are convergent and serve to define a function $\Delta w_{m+1}$ such that $d^k(\Delta_{m+1})/dt^k = \Delta w_{m+1}^{(k)}$, so that $\Delta w_{m+1} \in M_{n+k,n+k+2}$. Also, $(13.13_{m-1})$ implies

$$(13.18_m) \qquad (-1)^k \Delta w_{m+1}^{(k)} \leqq (1-\theta)\theta^{m-1} G \text{ for } t \geqq T,\ m > 0,$$

$$(13.19_m) \qquad |\Delta w_{m+1}^{(k+1)}| \leqq (\pi/\gamma_n)(1-\theta)\theta^{m-1} q^{\frac{1}{2}} G \text{ for } t \geqq T,\ m > 0.$$

By (12.9) and (13.17), the inequality $(13.18_m)$ implies $(13.7_m)$. Summarizing this paragraph,

$$(13.20) \quad (13.5_{m-1}),\ (13.5_m),\ (13.6_{m-1}),$$
$$(13.13_{m-1}) \text{ imply } (13.5_{m+1}),\ (13.6_m),\ (13.7_m).$$

It follows from (13.8), (13.12) and (13.20) that, in order to prove the existence of the sequence $w_0 \equiv 0, w_1, \cdots$ satisfying (13.5)-(13.7), it is sufficient to verify the existence of $w_1$ such that $w_1$ and $\Delta w_1 = w_1 - w_0 = w_1$ satisfy $(13.5_1)$, $(13.6_0)$ and $(13.7_0)$.

It is clear from (12.14)-(12.15) and $f_0 = F(t, 0, \cdots, 0)$ that $f_0 \in M_{n+1,0}$. Hence, by Theorem $6.1_n$, $(13.4_0)$ has a unique solution $w_1^{(k)}$ given by $(13.2_0)$ and $(-1)^k w_1^{(k)} \in M_{n,n+2}$. Also, by (6.3) and the remarks about it,

$$(13.21) \qquad 0 \leqq (-1)^k w_1^{(k)} \leqq \gamma_n f_0/q, \qquad |w_1^{(k)\prime}| \leqq \pi f_0/q^{\frac{1}{2}}$$

for $t \geqq T$. Thus (12.8), where $f = f_0$, shows that $(13.3_0)$ is meaningful for $j = 0, \cdots, k-1$ and defines a function $w_1 \in M_{n+k,n+k+1}$ with $d^k w_1/dt^k = w_1^{(k)}$. Also, $(13.3_0)$ and (12.8) imply $(13.7_0)$. This complete the induction.

Consequently, $w^{(j)}(t) = \lim w_m^{(j)}(t)$, as $m \to \infty$, exist uniformly for $j = 0, \cdots, k+1$ (hence, for $j = k+2, \cdots, k+n+2$ also) on compact subsets of $t \geqq T$. It follows that the limit function $w = w(t)$ is a solution of (12.11) for $t \geqq T$, is of class $M_{n+k}(T, \infty)$, and satisfies

$$(13.22) \quad (-1)^j w^{(j)}(t) \leqq \epsilon_{k-j}(t)/(k-j-1)! \text{ for } t \geqq T,\ j = 0, \cdots, k-1.$$

Also, by $(13.18_m)$ for $m \geqq 1$ and (13.21)

$$(13.23) \qquad (-1)^k w_m^{(k)}(t) \leqq G(t) + \gamma_n f_0(t)/q(t) \text{ for } m \geqq 0.$$

Thus, by Lebesgue's theorem on majorized convergence, it follows that, one can let $m$ tend to $\infty$ in $(13.3_m)$. In particular, (13.22) can be sharpened to

$$(-1)^j w^{(j)}(t) \leqq (-1)^k \int_t^\infty (s-t)^{k-j-1}[G(s) + \gamma_n f_0(s)/q(s)]ds/(k-j-1)!,$$

so that $w^{(j)} \to 0$ as $t \to \infty$ for $j = 0, \cdots, k-1$. By (12.4) and (12.15), $F(t, w(t), \cdots, (-1)^{k-1} w^{(k-1)}(t)) \in M_{n+1,0}(T, \infty)$. Thus if (12.11) is con-

sidered as a second order equation for $w^{(k)}$, it follows from Theorem 6.1$_n$ that $(-1)^k w^k \in M_{n,n+2}(T,\infty)$, hence $w \in M_{n+k,n+k+2}(T,\infty)$.

Since $T > 0$ is arbitrary for Theorem 12.3$_n$, the existence statement in that theorem $(n \leqq 2)$ is proved.

In order to complete the proof of the existence statement in Theorem 12.2$_n$, $n \leqq 2$, it is sufficient to verify the uniform convergence of the approximations $w_m(t)$ and their derivatives on closed subintervals of $0 < t \leqq T$. For the remainder of this section, assume that (12.11) is the equation (12.1) and that $0 < t_0 \leqq t \leqq T$.

By (13.13$_{m-1}$) and (13.15), it follows there exists a constant $C = C(t_0)$ such that

$$(13.24) \qquad |\Delta w_{m+1}^{(k)}| \leqq C\{\int_t^T |\Delta f_m(s)|\, ds + \theta\} \text{ for } m \geqq 1;$$

cf. the remarks following (7.5) for the estimate of $\int_T^\infty$ in (13.15). Repeated integrations of (13.24) over the interval $(t, T)$ and (13.7) give

$$(13.25) \qquad \begin{aligned} &|\Delta w_{m+1}^{(k-j)}| \\ &\leqq C\{\int_t^T (s-t)^{j-1} |\Delta f_m(s)|\, ds/(j-1)! + \theta^m \sum_{i=0}^{j} (T-t)^i/i!\}, \end{aligned}$$

for $j = 1, \cdots, k-1$ if $C$ is sufficiently large. Thus, by (13.9),

$$(13.26) \qquad \begin{aligned} &|\Delta f_{m+1}| \\ &\leqq C\{\int_T^t \sum_{j=0}^{k-1} (s-t)^{j-1} |\Delta f_m(s)|\, ds/(j-1)! + \theta^m \sum_{j=0}^{k-1} (T-t)^j/j!\}. \end{aligned}$$

In (13.26), $C = C(t_0)$ is a sufficiently large constant (independent of $m$). The existence of $C$ depends on the linearity of (12.1), so that $F_j^0$ in (13.9) is merely the coefficient function $g_j(t)$ in (12.1) and hence has a bound for $t_0 \leqq t \leqq T$ independent of $m$.

An induction on $m$ shows that

$$(13.27) \qquad |\Delta f_{m+1}| \leqq C \sum_{i=0}^{m-1} \theta^{m-1-i} (Ck)^i \sum_{j=i}^{\infty} (T-t)^j/j!$$

for $m \geqq 1$. Thus,

$$\sum_{m=1}^{\infty} |\Delta f_{m+1}| \leqq C(\sum_{i=0}^{\infty} \theta^i)(\sum_{i=0}^{\infty} (Ck)^i \sum_{j=i}^{\infty} (T-t_0)^j/j!) < \infty.$$

In view of (13.24)-(13.25), it follows that $w^{(j)} = \lim w_m^{(j)}$ exists uniformly for $0 < t_0 \leqq t \leqq T$ if $j = 0, \cdots, k$. Since an inequality similar to (13.24) holds for $\Delta w_{m+1}^{(k+1)}$, the uniform limit relation is valid for $j = k+1$; hence,

for $j = k + 2, \cdots, n + k + 2$ also. This completes the existence proof in Theorem 12. $2_n$, $n \leqq 2$.

**14. Uniqueness in Theorems 12. $2_n$ and 12. $3_n$.** The uniqueness assertion for $n = 0$ implies uniqueness for $n \geqq 0$. Thus it will be supposed that $n = 0$. In view of Theorem 6. $1_0$, it can be supposed that $k \geqq 1$.

Let $w = w(t)$ be the solution of (12.11) just constructed in the last section. Suppose that $w = W(t)$ is another solution with the stated properties. If $W(t)$ is compared with the 0-th approximations $w_0 = 0$, it is seen that the case $m = 0$ of

$$(14.1) \qquad (-1)^j (W^{(j)} - w_m^{(j)}) \geqq 0 \text{ for } j = 0, \cdots, k$$

holds. An induction shows the validity of (14.1) for $m = 0, 1, \cdots$. Thus

$$(14.2) \qquad (-1)^j (W^{(j)} - w^{(j)}) \geqq 0 \text{ for } j = 0, \cdots, k.$$

Let $T$ be so large that (13.22) holds and that the analogous inequalities hold for $w = W(t)$. Let

$$(14.3) \qquad \Delta w_m = W - w_m$$

and

$$(14.4) \qquad \Delta f_m = F(t, W, \cdots, (-1)^{k-1} W^{k-1}) - F(t, w_m, \cdots, (-1)^{k-1} w_m^{(k-1)}).$$

Then, the case $m = 0$ of

$$(14.5) \qquad (-1)^j \Delta w_m^{(j)} \leqq \theta^m \epsilon_{k-j}(t)/(k-j-1)! \text{ for } t \geqq T,\ j = 0, \cdots, k-1$$

holds. On repeating the arguments leading to (13. $7_m$), it is seen that (14.5) holds for $m \geqq 0$. Hence, $W - w = \lim \Delta w_m$, as $m \to \infty$, is 0 for $t \geqq T$. Since $W - w$ is a solution of a linear, homogenous differential equation for $t > 0$, it follows that $W - w = 0$ for $t > 0$. This proves the uniqueness assertion.

**15. Proof of Theorem 6. $1_n$, $q(\infty) < \infty$.** Let $n > 2$ and $0 < q(\infty) < \infty$ in the statement of Theorem 6. $1_n$. Let the differential equation (5.4) be differentiated $n - 2$ times to give

$$(15.1) \qquad w^{(n)} + q w^{(n-2)} + \sum_{j=0}^{n-3} C_{n-2,j} q^{(n-2-j)} w^{(j)} = f^{(n-2)}.$$

If this equation is identified with (12.1), then $k = n - 2$,

$$g_j = (-1)^{j+n-1} C_{n-2,j} q^{(n-2-j)}$$

and the $f$ on the right side of (12.1) is $(-1)^{n-2}f^{(n-2)}$. The assumptions of Theorem 6.$1_n$ imply those of Theorem 12.$1_2$ for $q$ and that $g_j$, $(-1)^{n-2}f^{(n-2)} \in M_{2+1,0}$. Clearly, condition (12.2) holds, and (12.3) hold when $q(\infty) < \infty$.

Since Theorem 12.$2_2$, has been proved, it follows that (15.1) has a unique solution $w = w(t) \in M_{n,n+2}$. Successive integrations of (15.1) show that $w = w(t)$ satisfies

$$w'' + qw = f + P_{n-3}(t), \tag{15.2}$$

where $P_{n-3}(t)$ is a polynomial of degree $n-3$. But since $q(\infty) < \infty$, $w \in M_{n,n+2}$ imply $w''$, $qw$, $f \to 0$ as $t \to \infty$, it follows that $P_{n-3}(t) = 0$. Hence, $w = w(t)$ is a solution of (5.4) with the desired properties. The uniqueness of this solution follows from the uniqueness of solutions of class $M_{n,n+2}$ for (15.1).

**16. Proof of Theorem 6.$1_n$.** It remains to prove the cases of Theorem 6.$1_n$, where $n > 2$ and $q(\infty) = \infty$.

By assumption, $q' \in M_n$. Hence $(-1)^{j+1}q^{(j)} \geqq 0$ for $j = 1, \cdots, n+1$ and $q^{(j)}(\infty) = 0$ for $j = 1, \cdots, n$. There exists a sequence of functions $h_1, h_2, \cdots$ on $t > 0$, such that $0 \leqq (-1)^n h_1 \leqq (-1)^n h_2 \leqq \cdots \leqq (-1)^n q^{(n+1)}$, $h_m \equiv 0$ for large $t$, and $q^{(n+1)}(t) = \lim h_m(t)$, as $m \to \infty$, uniformly on compact subsets of $t > 0$. Define $q_m(t)$ on $t > 0$ by

$$q_m{}^{(n+1)}(t) = h_m(t), \quad q_m{}^{(j)}(\infty) = 0 \text{ for } j = 1, \cdots, n \text{ and } q_m(1) = q(1);$$

so that, in particular,

$$q_m{}^{(j)}(t) = (-1)^{n-j+1} \int_t^\infty (s-t)^{n-j} h_m(s)\, ds/(n-j)! \text{ for } j = 1, \cdots, n$$

and $q_m' \in M_n$, $q_m(\infty) < \infty$. It is clear that $(-1)^{j+1}q_m{}^{(j)} \leqq (-1)^{j+1}q^{(j)}$ for $j = 1, \cdots, n+1$ and $t > 0$. Also

$$q_m{}^{(j)} \to q^{(j)} \text{ as } m \to \infty \tag{16.1}$$

uniformly on compact subsets of $t > 0$ for $j = 0, \cdots, n+1$. In particular, after discarding a finite number of $q_m$, if necessary, it can be supposed that $q_m(\infty) > 0$.

By the cases of Theorem 6.$1_n$ already proved, it follows that

$$w'' + q_m w = f \tag{16.2}$$

has a unique solution $w = w_m(t)$ of class $M_{n,n+2}$. If $G_m(t, s)$ is the Green's function belonging to (16.2), then

$$w_m(t) = \int_t^\infty G_m(t, s) f(s)\, ds. \tag{16.3}$$

By the proof of Theorem $6.1_0$ in Section 7, $q_m(t) > 0$ for $t \geqq T$ implies

$$(16.4) \qquad \left| \int_T^\infty G_m(t,s) f(s)\, ds \right| \leqq \pi f(T)/q_m(t).$$

It is clear from the uniformity of (16.1) for $j=0$ that $G_m(t,s) \to G(t,s)$, $m \to \infty$, uniformly on compact subsets of $s \geqq t > 0$. Hence (16.3), (16.4) imply that $w_m(t) \to w(t)$, $m \to \infty$, uniformly on compact subsets of $t > 0$; hence $w = w(t)$ is the solution (5.7) of (5.4). Similarly, it is shown that $w_m'(t) \to w'(t)$, $m \to \infty$, uniformly on compact subsets of $t > 0$.

It then follows from the differential equations (5.4), (16.2) and from (16.1) that $w_m^{(j)} \to w^{(j)}$, $m \to \infty$, uniformly on compact subsets of $t > 0$ for $j = 0, \cdots, n+3$. Thus $w_m \in M_n$ for $m = 1, 2, \cdots$ implies that $w \in M_n$. Since $w \in M_{0,2}$ by Theorem $6.1_0$, $w \in M_{n,n-1}$. Let (5.4) be differentiated $n-2$ times,

$$(16.5) \qquad w^{(n)} + q(t) w^{(n-2)} = -\sum_{j=0}^{n-3} C_{n-2,j} q^{(n-2-j)} w^{(j)} + f^{(n-2)}.$$

If the right side of (16.5) is multiplied by $(-1)^{n-2}$, it becomes a function of class $M_{3,0}$. Thus, considering (16.5) as an inhomogeneous differential equation of second order for $w^{(n-2)}$ with known right side, Theorem $6.1_2$ implies that there is a unique solution $w^{(n-2)} = w^{(n-2)}(t)$ such that $(-1)^{n-2} w^{(n-2)} \in M_{2,4}$. Uniqueness refers to the class of solutions satisfying $(w^{(n-2)})' \to 0$ as $t \to \infty$. Thus this solution $w^{(n-2)} = w^{(n-2)}(t)$ is the $(n-2)$-nd derivative of (5.7). Hence $w \in M_{n,n+2}$.

**17. Proof of Theorems $12.2_n$ and $12.3_n$, completed.** In view of Sections 13 and 16, it can be supposed that $k \geqq 1$ and $n > 2$. Let $w_0 \equiv 0$, $w_1, \cdots$ be the sequence of successive approximations defined in Section 13. By virtue of Theorem $6.1_n$, $n > 2$, proved in the last section, and a simple induction, it is seen that the $m$-th approximation $w_m$ is of class $M_{n+k,n+k+2}$ for $m = 0, 1, \cdots$. Furthermore, the convergences of the sequences $\{w_m\}, \{w_m'\}, \cdots, \{w_m^{(k+1)}\}$ proved in Section 13 implies the convergence of the sequences $\{w_m^{(k+2)}\}, \cdots, \{w_m^{(k+n+2)}\}$. Consequently $w_m \in M_{n+k}$ for $m = 0, 1, \cdots$ implies that $w = \lim w_m$ is of class $M_{n+k}$. Since $w \in M_{n+2,0}$ by Theorems $12.2_2$ or $12.3_2$, it follows that $w \in M_{n+k,n+k+1}$.

Differentiating (12.1) or (12.11) $n-2$ times and applying an argument similar to that at the end of the last section shows that $w \in M_{n+k,n+k+2}$.

## Part IV. Higher order monotony of $|z|^2$.

**18. The differential equation of Appell.** It was remarked by Appell ([1]; cf. [10], p. 298) that to a linear, homogeneous, second order differential equation, say $L_2u = 0$, there corresponds a linear, homogeneous, third order, differential equation $L_3w = 0$ such that if $u = x(t), y(t)$ are arbitrary solutions of $L_2u = 0$, then $w = x(t)y(t)$ is a solution of $L_3w = 0$. When $L_2u = 0$ is of the form

$$u'' + q(t)u = 0, \tag{18.1}$$

then $L_3w = 0$ is given by

$$w''' + 4q(t)w' + 2q'(t)w = 0. \tag{18.2}$$

An application of Theorem $12.1_n$ (and Remark 1) to (18.2) gives

THEOREM $18.1_n$. *Let $n \geqq 0$. Let $q(t)$ possess a derivative $q'(t)$ of class $M_{n+1}$ and $0 < q(\infty) < \infty$. Then (18.1) has a pair of solutions $u = x(t), y(t)$ such that*

$$w = x^2(t) + y^2(t) > 0 \tag{18.3}$$

*satisfies*

$$w(t) - 1 \in M_{n+1,n+3}. \tag{18.4}$$

*(The pair of solutions $(x, y)$ of (18.1) in (18.3) is unique up to their replacement by $(ax + by, cx + dy)$, where $a$, $b$, $c$, $d$ are constants such that $a^2 + c^2 = b^2 + d^2 = 1$, $ab + cd = 0$.)*

A corollary of this assertion and the theorem of Hausdorff-Bernstein is

COROLLARY. *Let $q(t)$ possess a derivative $q'(t)$ of class $M_n$ for $n = 1, 2, \cdots$ and $0 < q(\infty) < \infty$. Then (18.1) has a pair of solutions $u = x(t), y(t)$ such that (18.3) has a representation of the form*

$$w(t) = 1 + \int_0^\infty e^{-ts}d\sigma(s) \text{ for } t > 0 \tag{18.5}$$

*with a non-decreasing weight function $\sigma = \sigma(s)$.*

**19. Proof of Theorem $18.1_n$.** Let $u = u_1(t)$, $u_2(t)$ be linearly independent solutions of (18.1). Then the general solution of (18.2) is a linear combination of $u_1^2, u_1u_2, u_2^2$. It follows that if $w = w(t)$ is any solution of (18.2), then $w(t)$ can be written either in the form $w = \pm[x^2(t) + y^2(t)]$

or in the form $w = x^2(t) - y^2(t)$, where $u = x(t), y(t)$ are (possibly trivial) solution of (18.1).

The conditions of Theorem $18.1_n$ imply that (18.2) satisfies the conditions of Theorem $12.1_n$, where $k = 1$, $q$ is replaced by $4q$, $g_0(t) = 2q'(t)$ and $f(t) = 0$. Since $q(\infty) < \infty$, the derivative $q'(t)$ is integrable over $1 \leqq t < \infty$. Hence the monotony of $q'$ implies $g_0(\infty) = 0$. Thus, Theorem $12.1_n$ and the Remark 1 following it imply that (18.2) has a unique solution $w = w(t)$ satisfying (18.4). Because of the oscillatory nature of the solution of (18.1), this $w(t)$ must be of the form (18.3), where $u = x, y$ are linearly independent solutions of (18.1).

The uniqueness of the solution $w = w(t)$ of (18.2) implies the uniqueness of $x, y$ as specified. For the identity $x^2 + y^2 = (ax + by)^2 + (cx + dy)^2$ and the linear independence of the solutions $w = x^2, xy, y^2$ of (18.2) imply the given relations between the constants $a$, $b$, $c$, $d$.

**20. The case $q(\infty) = \infty$.** Theorem $18.1_n$ and its Corollary have analogues in the case $q(t) \to \infty$ as $t \to \infty$.

THEOREM $20.1_n$. *Let $n \geqq 0$. Let $q(t)$ possess a derivative $q'(t)$ of class $M_{n+1}$ and $q(\infty) = \infty$. Then* (18.1) *has a pair solutions $u = x(t), y(t)$ such that* (18.3) *satisfies*

$$w(t) \in M_{n+1, n+3}. \tag{20.1}$$

*(The uniqueness assertion of Theorem $18.1_n$ is valid.)*

The analogue of the Corollary of Theorem $18.1_n$ is

COROLLARY. *Let $q(t)$ possess a derivative $q'(t)$ of class $M_n$ for $n = 1, 2, \cdots$ and $q(\infty) = \infty$. Then* (18.1) *has a pair of solutions $u = x(t), y(t)$ such that* (18.3) *has a representation as a Laplace-Stieltjes integral*

$$w(t) = \int_0^\infty e^{-st} d\sigma(s) \quad \text{for } t > 0 \tag{20.2}$$

*with a non-decreasing weight function $\sigma = \sigma(s)$.*

**21. Proof of Theorem $20.1_n$.** The use of the Riemann-Liouville change of variables (2.7) in (18.1) leads to the differential equation (1.2), where $Q$ is given by (2.1). Note that $Q - 1 \geqq 0$ and

$$\int^\infty (Q - 1)\, ds = \int^\infty (Q - 1) q^{\frac{1}{2}}\, dt < \infty.$$

Thus, by a theorem of Bôcher (cf. [12], p. 261), (1.2) has a pair of solution $U = X(s), Y(s)$ which satisfy, in terms of the $s$-variable, as $s \to \infty$,

$$X(s) = \cos s + o(1), \qquad dX/ds = -\sin s + o(1),$$
$$Y(s) = \sin s + o(1), \qquad dY/d(s) = \cos s + o(1).$$

If $x(t) = q^{-\frac{1}{4}}(t)X(s)$, $y(t) = q^{-\frac{1}{4}}(t)Y(s)$ are the corresponding solutions of (18.1), then the boundedness of $q'$ and $ds = q^{\frac{1}{2}}dt$ imply that, as $t \to \infty$,

$$\begin{aligned} x(t) &= q^{-\frac{1}{4}}(t)(\cos s + o(1)), & x'(t) &= q^{\frac{1}{4}}(t)(-\sin s + o(1)), \\ y(t) &= q^{-\frac{1}{4}}(t)(\sin s + o(1)), & y'(t) &= q^{\frac{1}{4}}(t)(\cos s + o(1)), \end{aligned} \tag{21.1}$$

where $s = s(t) \to \infty$ as $t \to \infty$.

Consider the general solution $w = w(t)$ of (18.2),

$$w = ax^2 + bxy + cy^2$$

which can be written in the form

$$w = a[x^2 + y^2] + [bx + (c - a)y]y. \tag{21.2}$$

Thus, by (21.1), the derivative satisfies

$$w' = ao(1) + A[\sin(2s + \theta) + o(1)], \tag{21.3}$$

where $\theta = \theta(a, b, c)$ is a constant,

$$A = [b^2 + (c - a)^2]^{\frac{1}{2}} \geqq 0, \tag{21.4}$$

and the $o(1)$ terms have a monotone majorant, say, $\epsilon(t) \to 0$ as $t \to \infty$, independent of $a, b, c$. It follows that if $T$ is large and $w'(t)$ does not change signs on the interval $\frac{1}{2}T \leqq t \leqq T$, then

$$a \neq 0 \text{ and } 0 \leqq A \leqq |a| \epsilon(\tfrac{1}{2}T). \tag{21.5}$$

Suppose that for every large $T > 0$, there exists a function $q_T = q_T(t)$ on $t > 0$ with the properties that $q_T(t) \equiv q(t)$ for $0 < t \leqq T$ and that $q_T(t)$ satisfies the assumptions of Theorem 18.$1_n$ (that is, $q$ and $q_T$ satisfy the same conditions except that $0 < q_T(\infty) < \infty$ while $q(\infty) = \infty$).

By Theorem 18.$1_n$,

$$v'' + q_T(t)v = 0 \tag{21.6}$$

has a pair of solutions $v = x_T(t), y_T(t)$ such that $w_T = x_T{}^2 + y_T{}^2$ satisfies $w_T(t) - 1 \in M_{n+1,n+3}$. On $0 < t < T$, $v = x_T(t), y_T(t)$ are solutions of (18.1) and so, $w_T$ is of the form (21.2) on this interval. In particular, (21.2) is

of class $M_{n+1}(0, T)$ and, hence, (21.5) holds. After (21.2) is multiplied by a suitable positive constant, it can be supposed that $a = \pm 1$ and that (21.2) is of class $M_{n+1}(0, T)$.

In order to show the dependence on $T$, rewrite (21.2) as

$$w = w_T(t) = a_T x^2 + b_T xy + c_T y^2, \quad a_T = \pm 1. \tag{21.7}$$

Let $T$ tend to $\infty$ through a sequence of values for which $a = \lim a_T = \pm 1$ exists. Then it is clear from (21.4), (21.5) that

$$\lim_{T \to \infty} w_T(t) = a(x^2 + y^2) \tag{21.8}$$

and that this limit relation can be formally differentiated $n + 3$ times. If $S > 0$ is fixed, then $w_T \in M_{n+1}(0, S)$ if $T > S$. Hence $a = 1$ and the limit function $w = x^2 + y^2$ is of class $M_{n+1}(0, S)$ for every $S$, i.e., $w = x^2 + y^2 \in M_{n+1}$.

By (21.1), $w$ and $w'$ tend to 0 as $t \to \infty$. Hence, if (18.2) is considered as an inhomogeneous, second order equation

$$(-w')'' + 4q(t)(-w') = 2q'(t)w(t)$$

for $-w'$, then the (known) right side is of class $M_{n+1,0}$. Hence, by Theorem $6.1_n$, $-w' \in M_{n,n+2}$, that is, $w \in M_{n+1,n+3}$.

Thus, in order to complete the proof of Theorem $20.1_n$, it remains to verify the existence of the function $q_T(t)$ with the stated properties. Let $T > 0$ be large and fixed. Since it is desired that $q_T(t) \equiv q(t)$ for $0 < t \leqq T$, it suffices to define $q_T(t)$ on $t \geqq T$ so that $q_T' \in M_{n+1}(T, \infty)$, $q_T^{(j)}(T) = q^{(j)}(T)$ for $j = 0, \cdots, n+2$ and $q_T(\infty) < \infty$.

Note that for $j = 1, \cdots, n+1$

$$q^{(j)}(T) = (-1)^{n+j} \int_T^\infty (s-T)^{n+1-j} q^{(n+2)}(s)\,ds/(n+1-j)! \tag{21.9}$$

or, equivalently, for $j = 0, \cdots, n$,

$$(-1)^{n+j}(j!)\,q^{(n+1-j)}(T) = (-1)^{n+1} \int_0^\infty s^j dq^{(n+1)}(s+T). \tag{21.10}$$

But $q(\infty) = \infty$ implies

$$(-1)^{n+1} \int_0^\infty s^{n+1} dq^{(n+1)}(s+T) = \infty. \tag{21.11}$$

The relations (21.10) mean that the reduced moment problem for $j = 0, \cdots, n$,

$$\mu_j = \int_0^\infty s^j\, d\psi(s), \tag{21.12}$$

where $\mu_j = (-1)^{n+j}(j!)q^{(n+1-j)}(T)$, has a non-decreasing (absolutely continuous) solution $\psi(s) = (-1)^{n+1}q^{(n+1)}(s+T)$ for $s \geqq 0$. It is then clear from the conditions for the solvability of the Stieltjes' moment problem ([8], p. 6), that the finite sequence $\mu_0, \cdots, \mu_n$ can be extended to an infinite sequence $\mu_0, \mu_1, \cdots$ for which there exists a non-decreasing function $\psi(s)$ on $s \geqq 0$ satisfying (21.12) for $j = 0, 1, \cdots$.

In terms of such a solution $\psi$, define for $j = 1, \cdots, n+1$ and $t \geqq T$,

$$(21.13) \qquad q_T^{(j)}(t) = (-1)^{1+j} \int_t^\infty (s-t)^{n+1-j} d\psi(s)/(n+1-j)!$$

$$(21.14) \qquad q_T(t) = q(T) + \int_T^\infty q_T'(s)\,ds.$$

It is clear from (21.10) and the definitions of $\mu_0, \cdots, \mu_n$ that this definition of $q_T(t)$ for $t \geqq T$ together with the relation $q_T(t) = q(t)$ for $0 < t \leqq T$ give a function $q_T(t)$ on $t > 0$ such that $q_T'(t)$ is of class $M_{n-1}$ and $q_T(\infty) < \infty$. Since $\psi$ may not be continuous, $q_T$ need not have an $(n+1)$-st derivative. But it is clear that $(-1)^{n-1}q_T^{(n)}(t)$ is non-negative, non-increasing and convex. For the purposes of the proof of Theorem $20.1_n$ such a function $q_T(t)$ will suffice; cf. the remark following the Definition 6.1 of the class $M_n(a, b)$.

**22. The case of non-increasing *q*.** If $q$ is non-increasing, one has the following analogue of Theorem $20.1_n$.

THEOREM $22.1_n$. *Let $n \geqq 0$. Let $q(t)$ be non-increasing and let* (18.1) *be oscillatory at $t = \infty$; in particular, $q(t) > 0$. Let $1/q^2$ have a derivative of class $DM_{n+1}$. Then* (18.1) *has a pair of solutions $u = x(t), y(t)$ such that* (18.3) *satisfies*

$$(22.1) \qquad w'(t) \in DM_{n,n+2}$$

*and*

$$(22.2) \qquad w(t) \to 1 \text{ or } w(t) \to \infty, \text{ as } t \to \infty,$$

*according as $q(\infty) > 0$ or $q(\infty) = 0$. (The uniqueness assertion of Theorem $18.1_n$ is valid.)*

For the definition of the class $DM_n$, see Definition 6.3.

*Proof of Theorem $22.1_n$.* Divide (18.1) by $q$ and differentiate to obtain

$$(22.3) \qquad (u''/q)' + u' = 0.$$

Introduce the new variables

$$(22.4)\qquad U = u', \qquad d\tau = q(t)\,dt,$$

so that (22.3) becomes

$$(22.5)\qquad D^2U + (1/q)U = 0, \text{ where } D = d/d\tau = (1/q)\,d/dt.$$

As in Section 11, it is seen that $0 < t < \infty$ is mapped onto $(-\infty \leqq)T^0 < \tau < \infty$ and that (22.5) satisfies the assumptions of Theorem $20.1_n$.

Hence, (22.5) has a pair of solutions $U = X, Y$ such that

$$(22.6)\qquad W = X^2 + Y^2 > 0$$

and either

$$(22.7)\qquad W - 1 \in DM_{n+1,n+3} \text{ or } W \in DM_{n+1,n+3}$$

according as $q(\infty) > 0$ or $q(\infty) = 0$. Let $u = x, y$ be the solutions of (18.1) satisfying $X = x'$, $Y = y'$; cf. (22.4). Note that $W = x'^2 + y'^2$ and that, by (18.1) and (18.3)

$$-DW = -(2/q)(x'x'' + y'y'') = 2(x'x + y'y) \equiv w'.$$

Hence, (22.1) follows from (22.7).

The assertion that $w(t) \to 1$ as $t \to \infty$ in the case $q(\infty) > 0$ can be proved by the argument at the beginning of Section 20 involving the Riemann-Liouville change of variables.

When $q$ tends monotonously to 0 and (18.1) is oscillatory, (18.1) has at least one solution which is unbounded as $t \to \infty$; [4], p. 529(i). Hence $\limsup w(t) = \infty$ as $t \to \infty$. Since $w$ is monotone by the case $n = 0$ of (22.1), it follows that $w(t) \to \infty$ as $t \to \infty$ in the case $q(\infty) = 0$.

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# SYMMETRIC PRODUCTS AND JACOBIANS.*[1]

By Arthur Mattuck.

The $n$-fold symmetric product $C(n)$ of an algebraic curve $C$ is a variety closely related to the Jacobian variety $J$ of the curve. The low symmetric products appear birationally as a family of subvarieties of $J$ for which there is no good analogue on other abelian varieties and which have been used by Matsusaka to characterize Jacobians intrinsically among abelian varieties. The higher symmetric products are used to construct the Jacobian, either by the excisions-and-glue method of Weil, or the more precise projective method of Chow.

Now Chow's construction of the Jacobian as a quotient variety of $C(n)$ "fibered" by the linear systems [3] raises the question of whether $C(n)$, for $n > 2g-2$, is actually an algebraic projective bundle over $J$. We have shown elsewhere [8] that this is so; it is thus natural to ask what the Chern classes (to speak somewhat loosely) of this bundle are. One of the objectives of this paper is to exhibit these clases as elements of $A(J)$, the rational equivalence ring of $J$. Once this is done, one has according to a theorem of Grothendieck the structure of $A(C(n))$ explicitly as an extension of $A(J)$. This gives for example in a natural form the structure of the homology rings of high symmetric products of the closed orientable topological surfaces, which have hitherto only been computed "in principle" by the use of Eilenberg-MacLane spaces.

As a by-product of this determination of Chern classes, we get certain intersection relations among the subvarieties of $J$ alluded to above which can be thought of as generalizing to lower dimensions the well-known (and obvious) "relation" $\Theta = \Theta^*$; they express $W_i^*$ in terms of $W_i$.

The intersection relations on $C(n)$ and $J$ here given, in particular the basic formula of Section 7, have other applications. For example, they clarify and conceptually simplify the proofs of the intersection formulas for the $W_i$ given by Weil and Matsusaka [7] which play the crucial role in the characterization of $J$ by Matsusaka previously alluded to, also the formulas from

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which Weil's original proof of the Riemann hypothesis was derived. Again, they can be used as the basis for a geometric account of the Weierstrass points, with generalizations. These will appear separately.

Part II of this paper is devoted to the Chern classes. Part I is preliminary, and has connection with theorems of Chow and Andreotti. In it we prove that if $C$ is a non-hyperelliptic curve, then any $g-1$ points of a generic canonical divisor are algebraically independent (but as will be seen, only just!). In addition to being used in a critical argument of Part II, we also use it to squeeze out the dimension and irreducibility of certain subvarieties of $C(n)$ which play an important role in Part II, as well as in the other applications alluded to above.

Our emphasis throughout is on rational equivalence, not anything coarser.

**0. Preliminaries and notation.** We will work throughout over a fixed algebraically closed ground field $k$, and "generic" will always mean with respect to this field. All our basic varieties will be defined over $k$, and all points and divisors used in constructions unless specifically called generic, will be understood to be $k$-rational.

We denote by $C$ a fixed projective (complete) non-singular curve over $k$. Then $C(n)$ is its $n$-fold symmetric product, a non-singular projective variety most conveniently defined by the Chow coordinates [3, p. 456]. Its points represent the positive divisors of degree $n$ on $C$.

Except in Section 5, we shall reserve the word "divisor" exclusively for *non-negative zero-cycles on* $C$, and shall denote them by German letters $\mathfrak{a}, \mathfrak{b}, \cdots$. We will use a superscript to indicate their degree, wherever it is convenient in the argument to be reminded of it: thus $\mathfrak{a}$ and $\mathfrak{a}^r$ in the same context represent the same positive divisor of degree $r$. The dimension of a divisor, $\dim \mathfrak{a}$, always the geometric (projective) dimension of the complete linear system $|\mathfrak{a}|$ to which it belongs: so $\dim \mathfrak{a} = l(\mathfrak{a}) - 1$.

To avoid some tedious locutions, we shall often casually identify the point on $C(n)$ with the divisor of degree $n$ it represents, and thus speak of "the point $\mathfrak{a}^n$ on $C(n)$." Where we wish to be precise, we shall use $p(\mathfrak{a})$ for this point. Latin letters $x, y, p, q, \cdots$ will be reserved exclusively for points on $C$; capital letters for varieties and cycles on them. Superscripts and subscripts on capital letters in general refer to dimension and codimension, so that on $C(n)$, $X_i$ and $X^{n-i}$ represent the same cycle. Where indices must be used to distinguish cycles, they will conform to this convention.

If $X$ is a non-singular projective variety, by $A(X)$ we mean its rational equivalence ring graded by codimension, so that $A_i(X)$ is the group of cycle

classes of codimension $i$. An important formula we shall use constantly in the second part is the *projection formula*: Let $f\colon V \to U$ be a regular map, where $V$ is projective (more generally, let $f$ be proper). If $X$ is a cycle on $U$ such that $f^{-1}(X)$ is defined as a cycle, and $Y$ is a cycle on $V$ such that $Y \cdot f^{-1}(X)$ is defined, then in the sense of maps on cycles,

$$f(Y \cdot f^{-1}(X)) = f(Y) \cdot X,$$

the right side being automatically defined.

## Part I.

**1. The independence property.** Suppose fixed a linear system $\mathfrak{A}$ on the curve $C$, of degree $n$ and dimension $r$ (not necessarily complete), which we may think of as represented by the subvariety $A^r$ on $C(n)$ associated with it; a generic divisor of $\mathfrak{A}$ is then by definition one corresponding to a generic point of $A$. Recall that a positive divisor $\mathfrak{b}$ is said to be *contained in* a divisor $\mathfrak{a}$ if $\mathfrak{a} \geqq \mathfrak{b}$; it is contained in $\mathfrak{A}$ if it is contained in some $\mathfrak{a} \in \mathfrak{A}$. What can be said about the totality of positive divisors of some fixed degree $m$ contained in $\mathfrak{A}$? Their Chow points form a subset $A[m]$ of $C(m)$ about which we can say a little if we know that $\mathfrak{A}$ has the

*Independence property.* A linear system $\mathfrak{A}$ of dimension $r$ will be said to have the independence property if any $r$ points occurring in a generic divisor are independent generic points of $C$.

This notion has arisen incidentally in Chow's construction of the Jacobian [4], and in a weaker form (linear independence) in Andreotti's proof of Torelli's theorem [1, p. 813]. The property does not depend on the choice of generic divisor. We have then

THEOREM 1. *If the linear system $\mathfrak{A}$, of degree $n$ and dimension $r$ has the independence property, then*

(i) *$A[m]$ is a purely $r$-dimensional algebraic set on $C(m)$ if $m > r$, otherwise all of $C(m)$,*

(ii) *If $m \leqq r$, or nontrivially if $m \geqq n - r$, then $A[m]$ is even irreducible. This in particular will automatically be true (regardless of $m$) if $\mathfrak{A}$ is complete and either of degree $n > 2g - 2$, or the canonical system—always assuming it has the property.*

*Proof.* The points of $A[m]$ are those representing divisors $y_1 + \cdots + y_m$, where $y_1 + \cdots + y_m + \cdots + y_n \in \mathfrak{A}$ for suitable $y_i$ $(i > m)$.

Let $\mathfrak{a}=x_1+\cdots+x_n$ be a generic divisor of the system $\mathfrak{A}$. Then $\sum x_i\to\sum y_i$ is a specialization, since $A^r$ is irreducible and $\mathfrak{a}$ is generic; an extension of the specialization to the $x_i$ takes them in some order onto the $y_i$, so that $x_{i_1}+\cdots+x_{i_m}\to y_1+\cdots+y_m$ is a specialization for some for some choice of the $x_i$. In other words, every divisor represented by a point of $A[m]$ is a specialization of at least one of the ${}_nC_m$ divisors $x_{j_1}+\cdots+x_{j_m}$ of degree $m$ contained in $\mathfrak{a}$, and conversely, by similar reasoning it is clear that any specialization of one of these is represented by a point of $A[m]$.

Now by the independence property, any one of this finite number of divisors is either made up entirely of independent generic points (if $m\leqq r$) or contains $r$ of them (if $m\geqq r$), hence the locus of its specializations is respectively either $m$-dimensional (and therefore all of $C(m)$) or $r$-dimensional. The union of these loci is as we have seen $A[m]$, which proves statement (i) of the theorem.

To show the irreducibility of $A[m]$ if $m\geqq n-r$, it being $C(m)$ if $m\leqq r$ and therefore trivially irreducible, what we clearly must show is that any two of the ${}_nC_m$ divisors $x_{j_1}+\cdots+x_{j_m}$ are specializations of each other. Let therefore $\mathfrak{a}_1$ and $\mathfrak{a}_2$ be two such, so that

$$\mathfrak{a}=\mathfrak{a}_1+\mathfrak{b}_1=\mathfrak{a}_2+\mathfrak{b}_2,$$

where the $\mathfrak{b}_i$ are positive divisors of degree $n-m$, which is $\leqq r$ by hypothesis. By the independence property the $\mathfrak{b}_i$ are each made up of $n-m$ independent generic points, so that there is a specialization $\mathfrak{b}_1\to\mathfrak{b}_2$. Let $\mathfrak{a}_i'$ be a positive divisor of degree $r-(n-m)$ in $\mathfrak{a}_i$; then $\mathfrak{b}_i+\mathfrak{a}_i'$ is of degree $r$, has therefore only independent generic points, and so the specialization extends to $\mathfrak{b}_1+\mathfrak{a}_1'\to\mathfrak{b}_2+\mathfrak{a}_2'$. Extend it now to $\mathfrak{a}_1+\mathfrak{b}_1\to\mathfrak{c}$. Then $\mathfrak{c}$ is a divisor containing $\mathfrak{b}_2+\mathfrak{a}_2'$ and it is also a divisor of $\mathfrak{A}$, since it is a specialization of the generic divisor $\mathfrak{a}$. Since $A$ is of dimension $r$, and $\mathfrak{b}_2+\mathfrak{a}_2'$ has $r$ generic points, there can be only one divisor of $\mathfrak{a}$ containing $\mathfrak{b}_2+\mathfrak{a}_2'$, so that $\mathfrak{c}=\mathfrak{a}$. We have therefore a specialization

$$\mathfrak{a}_1+\mathfrak{b}_1\to\mathfrak{c}=\mathfrak{a}=\mathfrak{a}_2+\mathfrak{b}_2$$

extending $\mathfrak{b}_1\to\mathfrak{b}_2$, so that $\mathfrak{a}_1\to\mathfrak{a}_2$ is a specialization also, as was asserted.

As to the remaining statement, when will every positive $m$ between 0 and $r$ be either $\leqq r$ or $\geqq n-r$? If $n-r\leqq r+1$, that is, if $n\leqq 2r+1$. Now if the system $\mathfrak{A}$ is complete and of degree $n\geqq 2g-1$, then by the Riemann-Roch theorem, $r=n-g$, so indeed $n\leqq 2(n-g)+1$ while if $\mathfrak{A}$ is the canonical system, $n=2g-2$, $r=g-1$, and $2g-2\leqq 2(g-1)+1$: we even have room to spare!

**2. An independence criterion.** The following criterion results from analysis of a proof of Chow [4].

*A linear system* $\mathfrak{A}$ *of dimension* $r$ *has the independence property* $\Longleftrightarrow$ *for some choice of* $r-1$ *points* $y_1, \cdots, y_{r-1}$, *the linear system* $\mathfrak{A}-\sum y_i$ *has dimension one and no fixed points.*

Here $\mathfrak{A}-\sum y_i$ denotes the residual system (of degree $n-r+1$ if $\mathfrak{A}$ has degree $n$). Of course for general choice of the $y_i$ the system will always have dimension one, but it may have fixed points.

*Proof.* Let $x_1+\cdots+x_n$ be a generic divisor of $\mathfrak{A}$. Then the independence property is equivalent with the "generic" condition:

For every choice of $r-1$ independent generic points from among the $x_i$, the system $\mathfrak{A}-(x_{i_1}+\cdots+x_{i_{r-1}})$ has no fixed points.

Namely, this system is rational over $k(x_{i_1}, \cdots, x_{i_{r-1}})$, where $k$ is a field of definition for $C$ and $\mathfrak{A}$ over which the $x_{i_k}$ are independent generic points, and it has $(x_1+\cdots+x_n)-(x_{i_1}+\cdots+x_{i_{r-1}})$ as generic divisor. Thus a point of $C$ is a fixed point of this system if and only if it is one of the remaining $x_i$ and is algebraic over $k(x_{i_1}, \cdots, x_{i_{r-1}})$. Now if $\mathfrak{A}$ doesn't have the independence property, some $r$ of the $x_i$ are not independent, so we can indeed find such an algebraic $x_{i_r}$, and conversely the existence of such a situation means that $x_{i_1}, \cdots, x_{i_r}$ are not independent generic points, so that $\mathfrak{A}$ does not have the independence property.

This proves the forward implication of the criterion, since if $\mathfrak{A}$ has the property, one can take as the $y_i$ just the points $x_{i_1}, \cdots, x_{i_{r-1}}$. The implication is reversed by showing first that $\mathfrak{A}-\sum x_{i_k} \to \mathfrak{A}-\sum y_k$ is a specialization (viewing say the linear systems as irreducible cycles on $C(n-r+1)$). From this the theorem follows, for if the latter system has no fixed points, neither does the former—for example if you consider a finite set $\{\mathfrak{b}_i\}$ of divisors without common point from the second system, then a set of foreimages $\{\mathfrak{a}_i\}$ for them in the first system also can have no common point.

So relabel the independent generic points $x_{i_1}, \cdots, x_{i_{r-1}}$ as $x_1, \cdots, x_{r-1}$. Make a specialization $(x_1, \cdots, x_{r-1}) \to (y_1, \cdots, y_{r-1})$ and extend it to $\mathfrak{A}-\sum x_i \to \mathfrak{B}$, where $B$ will be a one-dimensional cycle on $C(n-r+1)$. We show the support of $B$ coincides with the one dimensional system $\mathfrak{A}-\sum y_i$ (this is enough for our theorem) by showing its points all represent divisors of the latter system. Indeed, any such point represents a divisor $y_r+\cdots+y_n$ that is, over the preceding specializations, itself a specialization of the generic divisor $x_r+\cdots, +x_n$ of the first system. Thus $x_1+\cdots+x_n \to y_1+\cdots+y_n$

is a specialization extending the preceding ones. Now since $x_1 + \cdots + x_n \in \mathfrak{A}$, so does $y_1 + \cdots + y_n$, so that $y_r + \cdots + y_n \in \mathfrak{A} - (y_1 + \cdots + y_{r-1})$, as was asserted.

## 3. Applications of the criterion.

(a). The following is due to Chow [4]. *If $n > 2g$, then every complete system $\mathfrak{A}$ of degree $n$ has the independence property.* Namely, the system has dimension $r = n - g$; let $x_1 + \cdots + x_n$ be a generic divisor, where the first $r$ points are independent generic, and choose for the $y_i$ the last $r - 1$ points. Then $\mathfrak{A} - \sum y_i = | x_1 + \cdots + x_{g+1} |$. But $n > 2g$ implies that $g + 1 \leqq r$, hence the $x_1, \cdots, x_{g+1}$ are independent generic and so this system has no fixed points, as required.

(b). THEOREM. *If $C$ is not hyperelliptic, then the canonical system $\mathfrak{W}$ has the independence property.* Take the $y_1, \cdots, y_{g-2}$ to be independent generic points, so that $\dim \mathfrak{W} - \sum y_i = 1$. We have to show this system has no fixed point $p$. If it did, then $\dim \mathfrak{W} - \sum y_i - p = 1$, so that by the Riemann-Roch theorem, $\dim | \sum y_i + p | = 1$. This is impossible however, because it is known that [1] the special divisors on $C(g-1)$—those belonging to linear systems of positive dimension—lie on a closed set whose dimension is less than $g - 2$, and which therefore cannot contain the $g - 2$ dimensional point which represents $y_1 + \cdots + y_{g-2} + p$. [Briefly, one considers the canonical mapping of $C(g-1)$ onto projective $g - 1$ space defined by the $g$ symmetric regular $g - 1$ forms on $C(g-1)$. If $C$ is not hyperelliptic, this map fails to be defined exactly where these differentials all vanish—in the hyperelliptic case it is always defined—and by direct calculation this occurs exactly at those points of $C(g-1)$ representing special divisors. Since $C(g-1)$ is nonsingular, this fundamental locus for the map must be of dimension $\leqq g - 3$.]

This result is of course false if $C$ is hyperelliptic.

(c). We do not need the following application in this paper, but have used it elsewhere to construct cross-sections of the projective bundle over the Jacobian [8].

PROPOSITION. *If $\mathfrak{A}$ is a complete linear system on $C$ of degree $n > 2g$, or if $\mathfrak{A}$ is the canonical system and $C$ is not hyperelliptic, then the rational map associated with $\mathfrak{A}$ is biregular, and the image $C'$ is projectively normal.*

*Proof.* The rational map we mean is the one turning the divisors of $A$ into hyperplane sections. Andreotti [1] has proved the second case of the

theorem; we prove the first case similarly (it is the one used in [8]). If $p_1$ and $p_2$ are two distinct points on $C$, then by the Riemann-Roch theorem, $\dim \mathfrak{A} - p_1 > \dim \mathfrak{A} - p_1 - p_2$; thus there is a divisor through $p_1$ not passing through $p_2$, $\mathfrak{A}$ separates points, and the map is one-one. Biregularity follows from the projective normality of the image, and this in turn follows by showing the linear system $\mathfrak{L}_k$ of hypersurface sections of degree $k$ is complete for all $k$, which we now do.

Let $\mathfrak{A}^{(k)}$ be the smallest linear system on $C$ containing all divisors of the form $\mathfrak{a}_1 + \cdots + \mathfrak{a}_k$, $\mathfrak{a}_i$ in $\mathfrak{A}$. Clearly $\mathfrak{A}^{(k)} \subset \mathfrak{L}_k$; we show $\dim \mathfrak{A}^{(k)} \geqq nk - g$, which will prove $\mathfrak{A}^{(k)}$ and therefore also $\mathfrak{L}_k$ is complete.

By the independence property, say, a generic divisor of $\mathfrak{A}$ contains no repeated points. Let $A$ and $B$ be non-overlapping sets containing respectively $n-g-1$ and $g$ of these points. Since $n \geqq 2g+1$, $B$ has never more than $n-g-1$ points (this many only when $n = 2g+1$). Since $\mathfrak{A}$ has dimension $n-g$ and the independence property, we can pass hyperplanes $H_1$ and $H_2$ through $A$ and $B$ respectively, each containing no other points of $\mathfrak{a}$. Add to them hyperplanes $H_3, \cdots, H_{k+1}$ not passing through any points of $\mathfrak{a}$. Then $H_1 + \cdots + H_{k+1}$ is a hypersurface of degree $k+1$ cutting out a divisor of $\mathfrak{A}^{(k+1)}$ and passing through the $n-1$ points of $A+B$, but not the $n$ points of $\mathfrak{a}$. Thus the $n$ points of $\mathfrak{a}$ must impose independent conditions on $\mathfrak{A}^{(k+1)}$, which shows that $\dim \mathfrak{A}^{(k+1)} \geqq \dim \mathfrak{A}^{(k)} - n$; since $\dim \mathfrak{A}^{(1)} = n-g$, the argument is complete.

In the sequel an important role will be played by a set of varieties $S^{(i)}$, $i = -1, 0, 1, \cdots$, which we now define.

*Definition.* $S^{(i)}$ is the set of all points on $C(g+i)$ representing special divisors $\mathfrak{a}^{g+i}$: those for which $\dim |\mathfrak{a}| > i$.

THEOREM 2. *$S^{(i)}$ is a variety of dimension $g-1$ for all $i = -1, \cdots, g-2$, and otherwise empty.*

*Proof.* Exactly those divisors $\mathfrak{a}^{g+i}$ are special which are contained in the canonical system, for by the Riemann-Roch theorem,

$$\dim |\mathfrak{a}| > i \Longleftrightarrow \dim |\mathfrak{W} - \mathfrak{a}| \geqq 0 \Longleftrightarrow \mathfrak{w} \geqq \mathfrak{a} \text{ for some } \mathfrak{w} \in \mathfrak{W}.$$

If $C$ is not hyperelliptic, by what we have proved, the canonical system has the independence property, and thus the result follows from Theorem 1.

If $C$ is hyperelliptic, we must argue directly. There is then, by definition of hyperelliptic, a regular map $f: C \to P^1$ of $C$ onto the projective line, of degree 2: that is, $[k(C) : k(P)] = 2$. The canonical system on $C$ is com-

posed of all divisors of the form $f^{-1}(x_1 + \cdots + x_{g-1})$, where $\sum x_i$ runs over the complete linear system of all positive divisors of degree $g - 1$ on $P$; namely, this is obviously a linear system, of degree $2g - 2$ and dimension $g - 1$, hence necessarily the canonical system.

Let now the $\{x_i\}$ be independent generic points, and let $f^{-1}(x_i) = y_i + y_i'$. Then the divisors of degree $g + i$ contained in the canonical system are all specializations of $y_1 + \cdots + y_{g-1} + y_1' + \cdots + y_{i+1}'$: this is trivial to see, but tedious to write out. The point on $C(g+i)$ representing this divisor is thus a generic point for $S^{(i)}$, which is therefore irreducible and of dimension $g - 1$.

## Part II.

**4. Some subvarieties of $C(n)$.** To avoid confusion, in this section only we shall use $p(\mathfrak{a})$ for the point on $C(n)$ representing the divisor $\mathfrak{a}$ on the curve $C$.

There is a natural map

$$f\colon C(r) \times C(n-r) \to C(n), \qquad 1 \leqq r \leqq n-1,$$

defined by $f[p(\mathfrak{a}^r), p(\mathfrak{b}^{n-r})] = p(\mathfrak{a} + \mathfrak{b})$. This map is regular, since it is single-valued and $C(r) \times C(n-r)$ is a normal variety. In fact, it is biregular on $p(\mathfrak{a}^r) \times C(n-r)$, as is easily seen.

We now define on $C(n)$ a subvariety denoted by $X[\mathfrak{a}]$ for all non-negative divisors $\mathfrak{a}$ as follows ($\phi$ denotes the empty set or divisor):

$$X[\phi] = C(n), \qquad X[\mathfrak{a}^n] = p(\mathfrak{a}), \qquad X[\mathfrak{a}^r] = \phi \text{ for } r > n,$$
$$X[\mathfrak{a}^r] = \text{image of } p(\mathfrak{a}^r) \times C\ (n-r) \text{ under } f\ (1 \leqq r \leqq n-1).$$

Thus if $r < n$, then $X[\mathfrak{a}^r]$ is biregularly equivalent to $C(n-r)$.

We are interested here in the intersection relations of these subvarieties given by the next two propositions.

Proposition 1. *If $\mathfrak{a}$ and $\mathfrak{b}$ have no common points, then $X[\mathfrak{a}]$ and $X[\mathfrak{b}]$ intersect properly on $C(n)$ and $X[\mathfrak{a}] \cdot X[\mathfrak{b}] = X[\mathfrak{a} + \mathfrak{b}]$.*

*Proof.* Set-theoretically we have under the assumptions evidently

$$X[\mathfrak{a}] \cap X[\mathfrak{b}] = X[\mathfrak{a} + \mathfrak{b}],$$

so that the intersection is proper. To show they intersect with multiplicity one, suppose first that $\mathfrak{a}$ and $\mathfrak{b}$ are independent generic divisors, so that neither $\mathfrak{a}$ nor $\mathfrak{b}$ has repeated points:

$$\mathfrak{a} = p_1 + \cdots + p_r \text{ and } \mathfrak{b} = q_1 + \cdots + q_s.$$

We use $C[n]$ to denote $C\times C\times\cdots\times C$ ($n$ factors), and consider the obvious regular map of degree $n!$

$$g\colon C[n]\to C(n)$$

defined by $g(x_1,\cdots,x_n)=x_1+\cdots+x_n$. We claim first of all that

$$g^{-1}(X[\mathfrak{b}])=\Sigma_\sigma q_1\times\cdots\times q_s\times C[n-s],$$

where the sum is over a set of $n!/(n-s)!$ permutations of the factors of $C[n]$ which make the summands on the right all distinct. In fact this relation is evidently true, set-theoretically, all coefficients must be the same for the summands on the right by reason of symmetry; if this coefficient is $m$, we have on applying $g$ to the left side $n!X[\mathfrak{b}]$, and applying it to the right side

$$\begin{aligned} m\,\Sigma_\sigma g\sigma(q_1\times\cdots\times q_s\times C[n-s]) \\ =\Sigma_\sigma(n-s)!X[\mathfrak{b}]=m[n!/(n-s)!](n-s)!X[\mathfrak{b}],\end{aligned}$$

whence $m=1$.

Now putting $Y[\mathfrak{a}]=p_1\times\cdots\times p_r\times C[n-r]$ on $C[n]$, we have (assuming $r+s\leqq n$, and defining $C[0]=\phi$)

$$Y[\mathfrak{a}]\cdot g^{-1}(X[\mathfrak{b}])=p_1\times\cdots\times p_r\times(\Sigma_\tau q_1\times\cdots\times q_x\times C[n-r-s]),$$

the sum taken over permutations $\tau$ which make the summands distinct. Applying $g$, we get therefore by the projection formula, since $g(Y[\mathfrak{a}])=(n-r)!X[\mathfrak{a}]$,

$$\begin{aligned} g(Y[\mathfrak{a}])\cdot X[\mathfrak{b}]=(n-r)!X[\mathfrak{a}]\cdot X[\mathfrak{b}] \\ =[(n-r)!/(n-r-s)!](n-r-s)!X[\mathfrak{a}+\mathfrak{b}],\end{aligned}$$

so that our result $X[\mathfrak{a}]\cdot X[\mathfrak{b}]=X[\mathfrak{a}+\mathfrak{b}]$ is proved.

If now $\mathfrak{a}'$ and $\mathfrak{b}'$ are non-generic divisors, specialize $(\mathfrak{a},\mathfrak{b})\to(\mathfrak{a}',\mathfrak{b}')$; this extends uniquely in turn to

$$\mathfrak{a}+\mathfrak{b}\to\mathfrak{a}'+\mathfrak{b}',\quad \mathfrak{a}\times C(n-r)\to\mathfrak{a}'\times C(n-r),\quad X[\mathfrak{a}]\to X[\mathfrak{a}'],$$

and so on. Thus since the intersection of specialized positive cycles is the specialization of the intersection (if all intersections are proper), we get $X[\mathfrak{a}']\cdot X[\mathfrak{b}']=X[\mathfrak{a}'+\mathfrak{b}']$.

PROPOSITION 2. *Let $\xi[\mathfrak{a}]$ denote the rational equivalence class in $A(C(n))$ of $X[\mathfrak{a}]$. Then $\xi[\mathfrak{a}]\cdot\xi[\mathfrak{b}]=\xi[\mathfrak{a}+\mathfrak{b}]$.*

*Proof.* If $\mathfrak{a}^r$ and $\mathfrak{b}^s$ have no common points, this is just a weakening of Proposition one. If they do, find finite sets of divisors $\{\mathfrak{a}_i{}^r\}$ and $\{\mathfrak{a}_j'{}^r\}$ having no points in common with $\mathfrak{b}$ such that $p(\mathfrak{a})\sim\Sigma\, p(\mathfrak{a}_i)-\Sigma\, p(\mathfrak{a}'_j)$, the

rational equivalence being on $C(r)$; it is a question only of avoiding certain subvarieties of $C(r)$ we shall not make explicit. Then

$$X[\mathfrak{a}] \sim \sum X[\mathfrak{a}_i] - \sum X[\mathfrak{a}_j']$$

on $C(n)$, since $X[\mathfrak{a}] = f(p(\mathfrak{a}) \times C(n-r))$ and rational equivalence is preserved by regular projective (proper) maps.

Now $\xi[\mathfrak{a}] \cdot \xi[\mathfrak{b}]$ is represented by

$$\begin{aligned}\sum X[\mathfrak{a}_i] \cdot X[\mathfrak{b}] - \sum X[\mathfrak{a}_j'] \cdot X[\mathfrak{b}] &= \sum X[\mathfrak{a}_i + \mathfrak{b}] - \sum X[\mathfrak{a}_j' + \mathfrak{b}] \\ &\sim X[\mathfrak{a} + \mathfrak{b}] \in \xi[\mathfrak{a} + \mathfrak{b}]\end{aligned}$$

since $p(\mathfrak{a} + \mathfrak{b}) \sim \sum p(\mathfrak{a}_i + \mathfrak{b}) - \sum p(\mathfrak{a}_j' + \mathfrak{b})$ on $C(r+s)$, this being so because rational equivalence is preserved by the map of $C(r) \times C(s) \to C(r+s)$.

**5. Chern classes of a projective bundle.** As general references for what follows, see [5, 10].

Suppose that $(E, X, \pi')$ is an algebraic vector bundle, that is, a fiber space in the sense of Andre Weil [12] whose fiber is a vector space $V^p$ of dimension $p$. In other words, the base space should be a non-singular variety covered by open sets $\{U_i\}$ such that $\pi'^{-1}(U_i)$ is biregularly isomorphic to $V^p \times U$ by a fiber-preserving isomorphism (local triviality), and such that the transition functions $g'_{ij}\colon U_i \cap U_j \to \mathrm{GL}(p, k)$ are regular maps into the general linear group. We may then consider the derived algebraic projective bundle $(P(E), X, \pi)$, the points of whose fibers $\pi^{-1}(x)$ are the lines through the origin in the vector space $\pi'^{-1}(x)$. Formally it is given by the transition functions $g_{ij}\colon U_i \cap U_j \to \mathrm{PGL}(p-1, k)$ derived from the natural homomorphism: $\mathrm{GL}(p, k) \to \mathrm{PGL}(p-1, k)$. Since each point of $P(E)$ represents a line, $P(E)$ is the base space of a canonically determined line bundle whose dual bundle is denoted by $L$; associated with $L$ is then a divisor class $\xi$ in $A_1(P(E))$.

The natural map $\pi^*$ is an isomorphism of $A(X)$ into $A(P(E))$; call its image $A(X)^*$ and write $c^*$ for $\pi^*(c)$, if $c \in A(X)$. Then Grothendieck has proved that

$$A(P(E)) = A(X)^*[\xi],$$

where the minimal equation for $\xi$ is:

$$\xi^p + c_1^* \xi^{p-1} + \cdots + c_p^* = 0.$$

The *Chern classes* of the vector bundle $E$ are now defined to be the $c_i$, and $1 + c_1 + \cdots + c_p$ is called the *total Chern class* of $E$. Note also that it follows from Grothendieck's result that $A_1(P(E)) = A_0(X)^* \cdot \xi + A_1(X)^*$.

So far we have started with the vector bundle and derived $\xi$ and the projective bundle from it. If we begin with a projective bundle, we may ask to what extent $\xi$ in the above is uniquely determined (or even exists).

PROPOSITION 3. *An algebraic projective bundle $(F, X, \pi)$ together with a given element $\xi \in A_1(F)$ is derived from a vector bundle $(E, X, \pi')$ as described above if and only if $\xi \cdot \pi^{-1}(x)$ is the class of a hyperplane in the projective space $\pi^{-1}(x)$, $x$ generic.*

*Proof.* For the necessity, since the restriction $L'$ of $L$ to a fiber $\pi^{-1}(x) = P^{p-1}$ is just the dual of the natural line bundle on $P$, we see that $L'$ is associated with the divisor class of a hyperplane section of $P$, from which it follows that $\xi \cdot \pi^{-1}(x)$ is in $A(\pi^{-1}(x))$ the generating element represented by a hyperplane in $\pi^{-1}(x)$.

For the sufficiency, Grothendieck [6, § 3.4] has shown that an algebraic projective bundle is always derived from a vector bundle; suppose therefore that our bundle $F$ is derived from $(E_0, X, \pi')$ with $\xi_0 \in A_1(F)$ as associated divisor class, so that $\xi_0 \cdot \pi^{-1}(x)$ is the class of a hyperplane in $\pi^{-1}(x)$. If now $\xi \in A_1(F)$ is any other element with this property, it follows from the structure of the group $A_1(F)$ as given above that $\xi = \xi_0 + \lambda_1{}^*$, where $\lambda_1 \in A_1(X)$. Let $L_1$ be the line bundle on $X$ associated with $\lambda_1$; then the vector bundle $E = E_0 \otimes L_1$ is the desired bundle: it has $\xi$ as associated divisor, and $E$ and $E_0$ have the same derived projective bundle, namely $F$.

Though this is all we need, for the sake of clarity we add a few remarks. It is easy to see that any vector bundle from which $F$ is derived is of the form $E \otimes L_1$, where $L_1$ is a line bundle on $X$. If now $X$ is complete, we get in this way a one-one correspondence between elements of $\xi + A_1(X)^*$ and vector bundles producing $F$. Now if $\xi$ as above is a root of the polynomial $f(X) = \sum_0^p c_i{}^* X^{r-i}$, then $\xi + \lambda_1{}^*$ is a root of $f(X - \lambda_1{}^*) = \sum d_i{}^* X^{p-i}$, whose coefficients are polynomials in $c_i$ and $\lambda_1$ (which are easily calculated) and in fact the Chern classes of $E \otimes L_1$. In other words, if we envision the elements $\lambda \in A_1(X)$ as acting on $A(X)$ as an additive group of automorphisms by

$$\lambda\colon 1 + c_1 + \cdots + c_n \to 1 + d_1 + \cdots + d_n \qquad (n = \dim X)$$

the $d_i$ being determined as above, then what is an invariant of a projective bundle is the orbit of $\sum c_i$ under the group $A_1(X)$: the elements of the orbit are in 1-1 correspondence with the Chern classes of the vector bundles from which $F$ is derived, and the orbit can reasonably be called the Chern class of $F$.

**6. Statement of the result.** We fix once and for all a point $p_0 \in C$. Then according to Section 4, we have a nested sequence of subvarieties of $C(n)$:

$$X[p_0] \supset X[2p_0] \supset \cdots \supset X[np_0]$$

which we shall abbreviate as $X_i = X[ip_0]$, so that $X_i$ is of codimension $i$ on $C(n)$. If we denote the rational equivalence class of $X_1$ by $\xi$, Proposition 2 shows that $X_i$ represents the class $\xi^i = \xi \cdot \cdots \cdot \xi$ ($i$ factors). Each $X_i$ is biregularly isomorphic to $C(n-i)$.

Now let $J$ be the Jacobian of $C$. Using our point $p_0$, we fix the canonical map $\phi\colon C \to J$ by making $\phi(p_0) = e$, the identity point of $J$. Then by linear extension of $\phi$ to divisors, we get a map

$$\pi\colon C(n) \to J.$$

We have proved elsewhere [8] that if $n > 2g-2$, which we shall henceforth assume, this triple $(C(n), J, \pi)$ is naturally a algebraic projective bundle whose fibers are the linear systems. On it we show now that the subvariety $X_1$ satisfies the hypotheses of Proposition 3: namely, for $x$ generic on $J$, $X_1 \cdot \pi^{-1}(x)$ is in the rational equivalence class of a hyperplane on $\pi^{-1}(x)$. For the divisors making up $X_1 \cap \pi^{-1}(x)$ are those containing $p_0$; these form a linear subsystem of dimension $n-g-1$, in other words, a hyperplane in the projective space to which $\pi^{-1}(x)$ is biregularly equivalent. To see that the intersection multiplicity is one, it is enough to show that (using Proposition 1), $X[\mathfrak{a}^{n-g-1} + p_0] \cdot \pi^{-1}(x) = X[\mathfrak{a}] \cdot X_1 \cdot \pi^{-1}(x)$ consists of a single point with multiplicity one, if $\mathfrak{a}$ is "general." In fact, it consists of the divisors of $\pi^{-1}(x)$ containing $\mathfrak{a}^{n-g-1} + p_0$; it is well known that there is only one, and moreover, since the linear system represented by the points of $\pi^{-1}(x)$ is rational over $k(x)$ [3, p. 475], this unique divisor will be $k(x)$-rational and hence its representative point on $C(n)$ will be $k(x)$-rational too. Under these circumstances, the intersection multiplicity at the point is one.

It follows therefore that $\xi$, the rational equivalence class of $X_1$ in $A_1(C(n))$ is associated with a unique vector bundle $E$ of rank $p = n - g + 1$ from which the bundle $C(n)$ is derived. We wish to compute the Chern classes of this bundle.

To this end, we let

$$W_i = \pi(X_{g+i})$$

so that $W_i$ for $0 \leqq i < g$ consists of all points on $J$ writable as $\phi(x_1) + \cdots + \phi(x_{g-i})$, and $W_g = e$, and $W_i = \phi$ for $i > g$. To eliminate asterisks, we also put

$$U_i = W_i{}^* = (W_i{}^-)_c,$$

that is, the transform of $W_i$ by the biregular map of $J$ sending $x$ into $-x+c$, where $c=\pi(\mathfrak{k}^{2g-2}+(n-2g+2)p_0)$ is the canonical point on $J$.

THEOREM 3. *As a cycle on $C(n)$, we have*

$$X_p-\pi^{-1}(U_1)\cdot X_{p-1}+\pi^{-1}(U_2)\cdot X_{p-2}-\cdots+(-1)^g\pi^{-1}(U_g)\cdot X_{p-g}=0,$$
$$p=n-g+1.$$

Letting $u_i$ be the rational equivalence class of $U_i$ in $A_i(J)$ and using the fact that the class of $X_i$ is $\xi^i$, we get immediately the

COROLLARY. *The total Chern class in $A(J)$ of the vector bundle $E$ with derived projective bundle $(C(n),J,\pi)$, $n>2g-2$, and associated $n-1$ cycle $X_1=X[p_0]$ is*

$$1-u_1+u_2-\cdots+(-1)^g u_g.$$

**7. Proof of Theorem 3.** In what follows, in addition to viewing points of $C(n)$ as divisors on $C$ without further comment, we shall often think of points on $J$ as divisor classes of degree 0 on $C$. Thus for example $\pi(np_0)=e$ and $\pi(\mathfrak{a})=\mathrm{Cl}(\mathfrak{a}-np_0)$, where $\mathrm{Cl}(\mathfrak{a})$ means the divisor class to which $\mathfrak{a}$ belongs. For reference, we state explicitly,

(1) The divisors in $X_i$ are those of the form $\mathfrak{a}^{n-i}+ip_0$.

(2) The classes in $U_i$ are those containing a representative of the form $\mathfrak{k}-\mathfrak{a}^{g-i}-(g+i-2)p_0$ for some $\mathfrak{a}$ ($\mathfrak{k}$ is a canonical divisor).

An essential auxiliary role is played by the $g-1$ dimensional varities $S^{(i)}$ representing the special divisors on $C(g+i)$ that we introduced at the end of Part I. Under the biregular isomorphisms of $C(g+i)$ onto $X_{n-(g+i)}$, the variety $S^{(i)}$ is carried onto a variety we shall continue to denote by $S^{(i)}$, so that

$$(3)\qquad S^{(i)}\subset X_{n-(g+i)},\qquad i=-1,\cdots,g-2,\cdots.$$

In particular, note the extreme cases: $S^{(-1)}=X_{n-g+1}=X_p$ since every divisor of degree $g-1$ is special and $S^{(g-1)}$ together with the higher $S^{(i)}$ are all empty.

We first note that set-theoretically,

$$(4)\qquad \pi(S^{(i)})=U_{i+2},\qquad i=-1,\cdots,g-2.$$

This shows, incidentally, that if $i<j\leqq g-2$, then $S^{(i)}\neq S^{(j)}$. To prove it, using (1) and (2) above, we have

$$\mathfrak{b}^{g+i}+(n-g-i)p_0\in S^{(i)}\Longleftrightarrow \mathfrak{b}^{g+i}\text{ special}$$
$$\Longleftrightarrow |\mathfrak{k}-\mathfrak{b}|\text{ contains }\mathfrak{a}^{g-i-2}\text{ for some }\mathfrak{a}$$

$$\Longleftrightarrow \mathfrak{k} - \mathfrak{a} \sim \mathfrak{b} \Longleftrightarrow \mathfrak{b} - (g+i)p_0 \sim \mathfrak{k} - \mathfrak{a} - (g+i)p_0$$

$$\Longleftrightarrow \mathrm{Cl}(\mathfrak{b} - (g+i)p_0) \text{ contains for some } \mathfrak{a} \text{ a divisor } \mathfrak{k} - \mathfrak{a}^{g-i-2} - (g+i)p_0$$

$$\Longleftrightarrow (\mathfrak{b} + (n-g-i)p_0) \in U_{i+2}.$$

Our main effort now goes into proving now the basic relation

$$\pi^{-1}(U_i) \cdot X_{p-i} = S^{(i-2)} + S^{(i-1)}, \qquad i = 1, \cdots, g, \quad p = n - g + 1. \tag{5}$$

From this our theorem follows easily. For writing (5) out for different $i$,

$$\pi^{-1}(U_1) \cdot X_{p-1} = X_p + S^{(0)} \qquad \text{since } X_p = S^{(-1)},$$
$$\pi^{-1}(U_2) \cdot X_{p-2} = S^{(0)} + S^{(1)}, \cdots,$$
$$\pi^{-1}(U_{g-1}) \cdot X_{p-g+1} = S^{(g-3)} + S^{(g-2)},$$
$$\pi^{-1}(U_g) \cdot X_{p-g} = S^{(g-2)} \qquad \text{since } S^{(g-1)} = \phi,$$

so that alternately adding and subtracting, we get

$$X_p = \pi^{-1}(U_1) \cdot X_{p-1} - \pi^{-1}(U_2) \cdot X_{p-2} + \cdots + (-1)^{g-1}\pi^{-1}(U_g) \cdot X_{p-g}$$

which is the desired relation.

*Proof of relation* (5). We first show the relation is true set-theoretically. The right hand side is clearly contained in the left, since if we remember that $X^i = X_{n-i}$ by our conventions, we have using (3),

$$S^{(i-2)} \subset X^{g+i-2} \subset X^{g+i-1}, \qquad S^{(i-1)} \subset X^{g+i-1} = X^{n-p+i} = X_{p-i}.$$

Also, by (4) we have since $W_{i+1} \subset W_i$,

$$\pi(S^{(i-2)}) = U_i, \qquad \pi(S^{(i-1)}) = U_{i+1} \subset U_i.$$

Looking at the reverse inclusion, we have using (1) and (2),

$$\pi(\mathfrak{a}^{g+i-1} + (n-g-i+1)p_0) \in U_i \Longleftrightarrow \mathfrak{a} \sim \mathfrak{k} - \mathfrak{b} + p_0 \text{ for some } \mathfrak{b}^{g-i}$$
$$\Longleftrightarrow \mathfrak{a} - p_0 \text{ is special.}$$

There are two possibilities: either $\dim \mathfrak{a} = \dim(\mathfrak{a} - p_0)$ or else $\dim \mathfrak{a} = 1 + \dim(\mathfrak{a} - p_0)$. In the first case, $p_0$ is a fixed point of $|\,\mathfrak{a}\,|$, so $\mathfrak{a} = p_0 + \mathfrak{a}^{g+i-2}$, where $\mathfrak{a}_0$ is special, so that $\mathfrak{a}_0 + (n-g-i+2)p_0$ is in $S^{(i-2)}$. In the second case, since $\dim(\mathfrak{a} - p_0) > i-2$ (speciality), we deduce $\dim \mathfrak{a} > i-1$, so that $\mathfrak{a}$ is special, hence $\mathfrak{a} + (n-g-i+1)p_0$ is in $S^{(i-1)}$.

We have finally to show the relation (5) is true as an intersection formula, that is, that the coefficients of the right side are both one. Suppose then that

$$\pi^{-1}(U_i) \cdot X_{p-i} = aS^{(i-2)} + bS^{(i-1)}, \qquad a, b > 0.$$

Let $\mathfrak{o}^{g-1}$ be a generic divisor of degree $g-1 = n-p$. Then by Proposition 1, $X_{p-i} \cdot X[\mathfrak{o}]$ is defined and equals $X[\mathfrak{o} + (p-i)p_0]$. If now we can show that

(i) the cycle $Z = \pi^{-1}(U_i) \cdot X[\mathfrak{o} + (p-i)p_0]$ is defined, $i = 1, \cdots, g$, it will follow by the associativity formula that

$$Z = aS^{(i-2)} \cdot X[\mathfrak{o}] + bS^{(i-1)} \cdot X[\mathfrak{o}].$$

This, combined with

(ii) $Z$ consists of points occurring all with multiplicity one,

(iii) $S^{(j)} \cap X[\mathfrak{o}]$ is not empty, $j = -1, \cdots, g-2$,

will then imply that both $a$ and $b$ are one, completing the proof of the theorem.

To prove statement (i), we have as before using (1) and (2),

$$\pi(\mathfrak{a}^i + \mathfrak{o} + (p-i)p_0) \in U_i \Longleftrightarrow \mathfrak{a} + \mathfrak{o} \sim \mathfrak{k} - \mathfrak{b}^{g-i} + p_0 \text{ for some } \mathfrak{b}$$
$$\Longleftrightarrow \mathfrak{a} + \mathfrak{b} \sim \mathfrak{k} + p_0 - \mathfrak{o}.$$

Now $|\mathfrak{k} + p_0|$ has dimension $g-1$, so that since $\mathfrak{o}^{g-1}$ is generic,

$$\dim |\mathfrak{k} + p_0 - \mathfrak{o}| = 0$$

and the system contains a unique positive divisor of degree $g$. Thus $\mathfrak{a} + \mathfrak{b}$ is a well-determined divisor of degree $g$, which means that $\mathfrak{a}$ must be one of the (in general) ${}_gC_i$ divisors contained in it. In other words,

$$\pi^{-1}(U_i) \cap X[\mathfrak{o} + (p-1)p_0]$$

consists of a finite set of points.

Going on to statement (ii) now, apply to (i) the map $\pi$ and use the projection formula, obtaining

$$\pi(Z) = U_i \cdot \pi(X[\mathfrak{o}^{n-p} + (p-i)p_0]) = U_i \cdot (W_{g-i})_x,$$

where $x = \pi(\mathfrak{o} + (n-g+1)p_0) = \pi(\mathfrak{o})$. It clearly is sufficient to show that $\pi(Z)$ has no multiple points, or by performing a translation by $-x$ on $J$ and remembering $U_i = (W_i)_c$, that $W_{g-i} \cdot (W_i^-)_{c-x}$ has no multiple points. However, A. Weil has computed this zero-cycle for us [12, Prop. 17, p. 74], the result being $\sum(w_{\alpha_1 \cdots \alpha_i})$, where the sum is taken over the ${}_gC_i$ combinations of indices $\alpha_1, \cdots, \alpha_g$ taken $i$ at a time, the points

$$w_{\alpha_1 \cdots \alpha_i} = \phi(q_{\alpha_1}) + \cdots + \phi(q_{\alpha_i})$$

and the $q_{\alpha_j}$ are defined by $c - x = \sum_1{}^g \phi(q_k)$; in these last two equations, $\phi$ is the canonical map of $C$ into $J$, and the sums are taken in the sense of the group law on $J$. Our job is therefore to see that the $w_{\alpha_1 \cdots \alpha_i}$ are all distinct.

The $q_k$ are determined by the above according to the relation $\mathfrak{k} - \mathfrak{o} + p_0 \sim \sum q_k$, so we have to show that no two of the divisors of degree $i$ contained in $(q_1 + \cdots + q_g)$ are linearly equivalent. The system $|\mathfrak{k} + p_0|$ has dimen-

sion $g-1$ and $p_0$ as fixed point. One divisor of this system is by the above $\mathfrak{o}+\sum q_k$; it is in fact a generic divisor (since $\mathfrak{o}$ is generic), and since $p_0$ is a fixed point of the system, say $p_0=q_g$. Then $\mathfrak{o}+(q_1+\cdots+q_{g-1})$ is a generic divisor of the canonical system $|\mathfrak{k}|$. But then the points $q_1,\cdots,q_{g-1}$ are independent generic points: in the non-hyperelliptic case because the canonical system has the independence property, while in the hyperelliptic case it is clear since a generic canonical divisor of $|\mathfrak{k}|$ is writable $\sum t_i+t_i'$, where $t_i$ and $t_i'$ are conjugates over a quadratic subfield of $k(C)$. There are thus three cases. Let $\mathfrak{a}^i$ and $\mathfrak{b}^i$ be two different divisors selected from $(q_1+\cdots+q_{g-1}+p_0)$. If both contain $p_0$ or if neither does, they cannot be linearly equivalent since, after subtracting the $p_0$ if necessary, two generic divisors of degree $\leqq g$ cannot be linearly equivalent, while if one contains $p_0$ but not the other, it is still impossible because a generic divisor of degree $i\leqq g$ has dimension zero and is therefore not linearly equivalent to another divisor of the same degree.

Finally to prove statement (iii), since the canonical system is of dimension $g-1$, we have (as in the preceding proof) a uniquely determined canonical divisor of the form $(q_1+\cdots+q_{g-1})+\mathfrak{o}^{g-1}$. Then clearly $\mathfrak{o}+(q_1+\cdots+q_{j+1})$ is a special divisor of degree $g+j$ (if $j=-1$, take just $\mathfrak{o}$), and so $S^{(j)}\cap X[\mathfrak{o}]$ cannot be empty because it contains

$$\mathfrak{o}+(q_1+\cdots+q_{j+1})+(n-g-j)p_0.$$

**8. The Euler characteristic.** Out of curiosity, and in order to sneak Newton into this paper, we compute the Euler characteristic of the vector bundle whose Chern classes we have just determined in Theorem 3.

The Chern classes of $J$—that is, the Chern classes of the tangent bundle to $J$—are trivial, since the tangent bundle is trivial: it is enough to show that the dual bundle of 1-forms is trivial, but this is evident because it has a basis at every point consisting of the $g$ linearly independent invariant regular simple differentials on $J$. The Riemann-Roch-Hirzebruch formula [2] thus reduces to

$$\chi=\kappa_g(e^{\delta_1}+\cdots+e^{\delta_p}),$$

where the $\delta_i$ are defined formally, by the Corollary to Theorem 3, by

$$1-u_1x+u_2x^2-\cdots(-1)^g u_g=(1+\delta_1 x)\cdots(1+\delta_g x),$$
$$\delta_i=0,\ i>g.$$

Expanding the exponential series and looking just at the term of weight $g$, we get

$$\chi=\kappa_g[(1/g!)(\delta_1^g+\cdots+\delta_g^g)].$$

Letting $s_k = \delta_1{}^k + \cdots \delta_g{}^k$, we now invoke, for $k = 1, \cdots, g$, [9],

*Newton's identities:* $s_k + u_1 s_{k-1} + \cdots + u_{k-1} s_1 = -k u_k$.

These give a system of linear equations for determining the $s_k$, whose solution for $s_g$ is by Cramer's rule,

$$s_g = \begin{vmatrix} -g u_g & u_1 & u_2, \cdots, u_{g-1} \\ -(g-1) u_{g-1} & 1 & u_1, \cdots, u_{g-2} \\ \cdot & \cdot & \cdot \quad \cdot \\ \cdot & \cdot & \cdot \quad \cdot \\ -u_1 & \cdot & \cdot \quad 1 \end{vmatrix}.$$

According now to intersection formulas of Matsusaka and Weil modulo numerical equivalence [7], $u_1{}^m \equiv m! u_m$, from which we deduce, if $\sum i_k = g$,

$$u_{i_1} u_{i_2} \cdots u_{i_r} \equiv (u_1{}^{i_1}/i_1!) \cdots (u_1{}^{i_r}/i_r!) \equiv g! u_g / i_1! \cdots i_r!.$$

All the terms in the above determinant, when it is expanded out, are indeed of weight $g$, and we get therefore

$$\chi = \kappa_g[(1/g!) s_g] = \begin{vmatrix} -g(1/g!) & 1 & \tfrac{1}{2}! \cdots 1/(g-1)! \\ -(g-1)/(g-1)! & 1 & 1 \cdots 1/(g-2)! \\ \cdot & & \\ \cdot & & \\ -1 & \cdot & \cdot \cdot \cdot \cdot \quad 1 \end{vmatrix}.$$

Thus $\chi = 0$, since the first and last columns differ by a sign.

**9. Some relations in $A(J)$.** Quite generally, if $(P(E), X, \pi)$ is a projective bundle derived from a vector bundle $E$ of rank $p$ over a base space $X$ of dimension $g$ and $\xi \in A_1(P(E))$ is the associated divisor class, we can deduce trivially from the Chern relation

$$\xi^p + c_1{}^* \xi^{p-1} + c_2{}^* \xi^{p-2} + \cdots + c_p{}^* = 0$$

some relations in $A(X)$. Namely, multiply the relation through by $\xi^i$, $i = 0, \cdots, g-1$, project onto $X$ and use the projection formula; this gives

$$\pi(\xi^{p+i}) + c_1 \pi(\xi^{p+i-1}) + \cdots + c_p \pi(\xi^i) = 0 \qquad i = 0, \cdots, g-1,$$

a set of relations which may be summarized as

$$(1 + c_1 + c_2 + \cdots + c_p)(1 + \pi(\xi^p) + \pi(\xi^{p+1}) + \cdots + \pi(\xi^{p+g-1})) = 1,$$

in view of the fact that $\pi(\xi^{p-1}) = 1$, $\pi(\xi^j) = 0$ if $j < p-1$.

Applying this to our situation, we have $\pi(\xi^{p+i}) = w_{i+1}$, and so

$$(1 - u_1 + u_2 - \cdots + (-1)^g u_g)(1 + w_1 + w_2 + \cdots + w_g) = 1;$$

written in terms of the cycles, this becomes

THEOREM 4. *If $w_i$ is the rational equivalence class of $W_i$ (in Weil's notation $W_{g-i}$) on $J$ and $u_i$ the class of $W_i^* = (W_i^-)_c$, then*

$$w_1 - u_1 = 0,$$
$$w_2 - u_1 w_1 + u_2 = 0, \cdots,$$
$$w_g - u_1 w_{g-1} + \cdots + (-1)^g u_g = 0.$$

These express therefore the $u_i$ in terms of the $w_i$. The nature of the relations suggest that if one applies the map $\sigma: x \rightarrow -x + c$ to $J$, the resulting projective bundle $(C(n), J, \sigma\pi)$ whose Chern classes are $(-1)^i w_i$—or rather, prehaps its dual—should be in some sense the "opposite" bundle to $(C(n), J, \pi)$.

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CORRECTION TO "APPLICATIONS OF THE THEORY OF MORSE TO SYMMETRIC SPACES" (This Journal, vol. 80 (1958), pp. 964-1029).*

By Raoul Bott and Hans Samelson.

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It has been pointed out to us by H. Seifert that our characterization of simplices "hanging over critical points" in the paper cited above (conditions (a) (b) (c) on p. 978) is insufficient and that therefore our description of the theory of Morse in Proposition 8.3, p. 978, is incorrect as stated. If the space $\Omega$ were a manifold, one would only have to add the requirement that the singular simplices in question be differentiable and non-degenerate (have non-singular differential, at least at the barycenter). But in a function space the description of the associated simplices is more complicated. One of Morse's procedures (cf. [16, pp. 38, 56, 58; 17, pp. 52, 81]) is to introduce a set of cross manifolds $Q_k$ (including the end-manifold) along the geodesic segment $s$, and with their help to imbed the point $s$ of $\Omega$ into a manifold $P_s = \prod_k Q_k$ of geodesic polygons, contained in $\Omega$. The function $L \mid P_s$ is $C^\infty$ and has a non-degenerate singularity of index $\lambda_s$ at $s$. The main fact is now that any singular simplex associated to $s$ in $P_s$ also serves as associated simplex in $\Omega$. With this in mind we shall prove below that the singular simplex $(\sigma_s, \phi_s)$ of p. 984, l. 27 is indeed associated to $s$. (This is where we had used our incorrect characterization.) The remainder of the proof on pp. 984, 985 is unchanged. Proposition 10.2(d), p. 981, and its proof in Section 11, pp. 983, 984, become unnecessary.

We may and shall assume that among the $Q_k$ there is one at each exceptional point $s(t_i)$ and one at some point $s(\bar{t}_i)$ with $\bar{t}_i$ in $(t_i, t_{i+1})$ for $i = 1, \cdots, n$; further each cross manifold contains locally the orbit of the point to which it is attached.

The map $f_s\colon \Gamma_s \rightarrow \Omega$, restricted to a small neighborhood of $\omega_s$, factors then through a $C^\infty$-map $h_s$ into $P_s$.

Lemma (a). *$h_s$ is non-degenerate at $\omega_s$.*

*Proof.* Let $X$ be a non-zero vector of $\Gamma_s$ at $\omega_s$; let $Y$ be a vector of $W_s$ at $(e, \cdots, e)$ with $\dot{\psi}_s(Y) = X$. We write $Y = (Y_1, \cdots, Y_n)$, with $Y_j$ a vector of $K_j$. Let $Q$ be a cross manifold, attached to some $\bar{t}_i$. The image of $X$ under the composition of $h_s$, projection of $P_s$ onto its factor $Q$, and inclusion of $Q$ in $M$ is easily found to be $(\tilde{Y}_1 + \cdots + \tilde{Y}_i)(s(\bar{t}_i))$ (cf. def. 1.6 for $\tilde{Y}$).

* Received December 20, 1960.

If $i$ is the smallest value for which $Y_i$ is not tangent to the subgroup $K_s$ of $K_i$ (this exists since $K \neq 0$), the vector so obtained is not zero, since $\bar{t}_i$ is not exceptional, and Lemma (a) is proved.

Next we consider a local lemma. Let $f$ be a $C^\infty$-function on a neighborhood $U$ of a point $p$ in a manifold of dimension $m$, with $p$ as only critical point, non-degenerate, of index $\lambda$. Write $A$ for $\{x \in U : f(x) \leqq f(p)\}$, $A^-$ for $\{x \in U : f(x) < f(p)\}$, $A_0$ for $\{x \in U : f(x) = f(p)\}$.

Let $\mu : \sigma \to U$ be a $C^\infty$-singular simplex of dimension $\lambda$ such that

(i) $\mu^{-1}(p)$ consists of just the barycenter $b$ of $\sigma$,

(ii) $f \circ \mu$ is constant, i. e., $\mu(\sigma) \subset A_0$,

(iii) the differential of $\mu$ is non-singular at $b$.

LEMMA (b). *Under the above hypothesis, $\mu$ represents a generator of $H_\lambda(A, A - \{p\})$, for any coefficient group. Further, if $\mu' : \sigma \to U$ is a singular simplex sufficiently close to $\mu$, with $\mu'(\sigma) \subset A^- \cup \{p\}$, then $\mu'$ is a generator of $H_\lambda(A^- \cup \{p\}, A^-)$, again for any coefficient group, i. e. it serves as associated simplex for the critical point $p$.*

*Proof.* We may assume (by [16]) that $U =$ Euclidean space $E^m$, that $p$ is the origin and that $f$ has the form $-x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_m^2$. Both pairs $(A, A - \{p\})$ and $(A^- \cup \{p\}, A^-)$ have as deformation retract the pair $(E^\lambda, E^\lambda - \{p\})$, where $E^\lambda$ is th subspace of $E^m$ spanned by the first $\lambda$ axes; the retraction map is identical with the projection $\theta$ along the orthogonal complement of $E^\lambda$. Because of (iii) the "light cone" $A_0$ contains linear subspaces of dimension $\lambda$; any such space maps in non-degenerate fashion under $\theta$. The differential of $\theta \circ \mu$ is therefore non-degenerate at $b$, and the lemma follows by standard arguments. Incidentally, $\lambda$ is necessarily $\leqq m/2$.

We come now to the proof that the simplex $(\sigma_s, \phi_s)$ of no. 12, p. 984, is associated to $s$. The simplex $h_s \circ \rho_s : \sigma_s \to P_s$ (defined if $\rho_s(\sigma_s)$ is small enough) satisfies the hypotheses of Lemma (b); (i) is clear, (ii) follows from the second sentence on p. 981, and (iii) is implied by Lemma (a). The standard retraction of a neighborhood of $s$ (in $\Omega$) into $P_s$ sends the singular simplex $\phi_s$ into a singular simplex $\phi_s'$ with $\phi_s'(\sigma_s) \subset P_s$. Since the $L$-value does not increase under the retraction, Lemma (b) applies, so that $\phi_s'$, being close to $h_s \circ \rho_s$ for small $u$, is associated to $s$ in $P_s$; but then $\phi_s$ is associated to $s$ in $\Omega$. Q. E. D.

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## ON THE PREPARATION OF MANUSCRIPTS.

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Neither a solidus nor a negative exponent is needed in the symbols $\frac{1}{2}$, $\frac{1}{\pi}$, $\frac{1}{2\pi}$, $\frac{1}{2\pi i}$, which are available in regular size type.

Binomial coefficients should be denoted by $C_k{}^n$ and not by parentheses. Correspondingly, for symbols of the type of a quadratic residue character the use of some non-vertical arrangement is usually imperative.

For square roots use either the exponent $\frac{1}{2}$ or the sign $\surd$ without the top line, as in $\surd -1$ or $\surd(a+b)$.

Replace $e^{(\ )}$ by exp( ) if the expression in the parenthesis is complicated.

By an appropriate choice of notations, avoid unnecessary displays.

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Use $'$ or $d\ /dx$, possibly $D$, but preferably not a dot, in order to denote ordinary differentiation and, as far as possible, a subscript in order to denote partial differentiation (when the symbol $\partial$ cannot be avoided, it should be used as $\partial\ /\partial x$).

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In a determinant use a notation which reduces it to the form det $a_{ik}$.

Subscripts and superscripts cannot be printed in the same vertical column, hence the manuscript should be clear on whether $a_i{}^k$ or $a^k{}_i$ is preferred. (Correspondingly, the limits of summation must not be typed *after* the $\Sigma$-sign, unless either $\Sigma_1{}^m$ or $\Sigma^m{}_1$ is desired.) If a letter carrying a subscript has a prime, indicate whether $x_i{}'$ or $x'{}_i$ is desired.

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Distinguish carefully between l. c. "oh," cap. "oh" and zero. One way of distinguishing them is by underlining one or two of them in different colors and explaining the meaning of the colors.

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Avoid unnecessary footnotes. For instance, references can be incorporated into the text (parenthetically, when necessary) by quoting the number in the bibliographic list, which appear at the end of the paper. Thus: "[3], pp. 261-266."

Except when informality in referring to papers or books is called for by the context, the following form is preferred:

[3] O. K. Blank, "Zur Theorie des Untermengenraumes der abstrakten Leermenge," *Bulletin de la Societé Philharmonique de Zanzibar*, vol. 26 (1891), pp. 242-270.

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